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MATHEMATICS-I

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SYLLABI-BOOK MAPPING TABLE

Mathematics-I

Syllabi	Mapping in Book
Trigonometry: System of measuring angles, Trigonometric functions, Identities and signs, Values of t-ratio for t-ratios of allied angles, Addition and subtraction formulae, Transformation of products into sum or difference of t-ratios, Transformation of sum or difference into product of t-ratios, Trigonometric equations and graphs, Inverse trigonometric function.	**Unit 1:** Trigonometry **(Pages: 3–99)**
Differentiation: Elementary results on limits and continuity (without proof). Derivative of functions, Differentiation of implicit functions and parametric forms.	**Unit 2:** Differentiation **(Pages: 101–157)**
Coordinate Geometry: Distance formulae, Section formulae, Slope of non-vertical line, Equation of line slop-intercept form, Normal form, Distance of a point from a line, Angle between two lines.	**Unit 3:** Coordinate Geometry **(Pages: 159–183)**
Quadratic Equations: Solution of quadratic equations by factor method, Complete square method, and discriminant method, Relation of the roots.	**Unit 4:** Quadratic Equations **(Pages: 185–221)**
Complex Numbers: Definition, Representation of complex number, Argand plan, Sum, Subtraction, product and division of complex numbers, Magnitude, argument and square root of complex numbers.	**Unit 5:** Complex Numbers **(Pages: 223–262)**

CONTENTS

INTRODUCTION

NOTES

Mathematics is taught as a core subject in almost all undergraduate courses in applied sciences, especially management and computer science. It is now generally accepted that a study of any course is incomplete without the knowledge of mathematics. Due to unfounded fear of the so called 'difficult' mathematics, students tend to shy away from the subject and often, it becomes difficult to persuade them to take special courses in mathematics. However, the fact is that learning to use mathematical techniques does not require elaborate mathematical preparation and the student need not be dismayed by some mathematical manipulations. One learns through practice. Mathematics is best learnt through systematic learning. The learner is bound to find it rewarding and exciting, while following the logical arguments and steps involved.

Mathematics-I is a suitable textbook for the students of BCA. Topics, such as trigonometry, differentiation, coordinate geometry, quadratic equations and complex numbers, have been covered in detail in this book.

This book gives a simple and clear presentation of mathematics which will be useful to beginners. There is a fairly self-contained development of the topics and the student will find here a starting point which will help gain self-confidence and familiarity with the subject. While the knowledge of elementary mathematics is assumed, an attempt has been made to explain simple terms also. The student will attain proficiency in mathematics, while he proceeds further with different aspects of his studies.

The book follows the self-instructional mode wherein each unit begins with an Introduction to the topic. The Unit Objectives are then outlined before going on to the presentation of the detailed content in a simple and structured format. Check Your Progress questions are provided at regular intervals to test the student's understanding of the topics. A Summary, a list of Key Terms and a set of Questions and Exercises are provided at the end of each unit for recapitulation.

UNIT 1 TRIGONOMETRY

NOTES

Structure

1.0 INTRODUCTION

Trigonometry is that branch of Mathematics which deals with the measurement of angles. It is derived from two Greek words '*trigonon*' (a triangle) and '*metron*' (a measure), thereby meaning measurement of triangles. However, now this definition has been modified to include measurement of angles in general, whether the angles are of a triangle or not. In this unit, you will learn about plane trigonometry which is restricted to the measurement of angles in a plane.

Trigonometry is related to the calculation of sides and angles of triangles with the help of trigonometric functions. The most familiar trigonometric functions are the sine, cosine and tangent. Trigonometric functions were primarily employed in mathematical tables.

In this unit, you will learn to measure angles and use trigonometric functions. In addition, you will be introduced to signs and identities along with the standard ratios of commonly used angles. Finally, this unit will deal with the complex trigonometric calculations and computing.

1.1 UNIT OBJECTIVES

NOTES

After going through this unit, you will be able to:

- Understand the various ways of measuring angles in a plane
- Explain the various trigonometric functions and use these applications to solve problems related to angles
- Understand the concepts of identities and signs
- Discover and prove the trigonometric ratios of the standard as well as allied angles
- Understand the various features of inverse trigonometric functions
- Comprehend and resolve trigonometric equations
- Modify trigonometric ratios of basic computing

1.2 SYSTEM OF MEASURING ANGLES

1.2.1 Angles

An angle is defined as the rotation of a line on one of its extremities in a plane from one position to another. Two lines are said to be at *right angles*, if a revolving line starting from one position to another describes a quarter of a circle. In case the revolving line moves in anticlockwise direction, then the angle described by it is positive, else, it is negative. There is no limitation to the size of angles in trigonometry.

Consider the coplanar lines $X'OX$ and YOY' at right angles to each other (see Figure 1.1). Now, if the revolving line starts from OX and reaches the position OA, in the anticlockwise direction, then it is described as a positive angle that is less than a right angle. If it continues to move in the same direction, then in the position OB, it has an angle, XOB, which is more than a right angle. In the position OC, which is it has described an angle XOC, which is more than two right angles, but less than three right angles. Similarly, in the position OD, it has described an angle XOD less than four right angles but more than three right angles. Now, if it continues to revolve in the same direction, then in the position OA, it has described an angle equal to four right angles along with $\angle XOA$. In this way, an angle of any size can be described. Thus, there is no limitation to the size of angles that can be described in trigonometry.

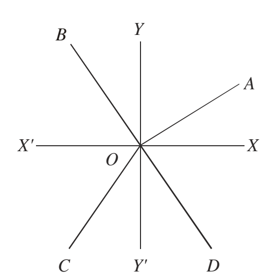


Figure 1.1 *Angles*

Quadrants

The four portions XOY, $X'OY$, $X'OY'$ and XOY' into which a plane is divided are called *first*, *second*, *third* and *fourth quadrants* respectively.

NOTES

1.2.2 Measurement of Angles

To measure angles, a particular angle is fixed and is taken as a unit of measurement, so that any other angle is measured by the number of times it contains the unit. For example, if $\angle XOY$ is a right angle, and it is taken as a unit of measurement, then $\angle XOX'$ is equal to two right angles, as it contains two units.

You will study the following three systems of measurement:

1. Sexagesimal System (English System)
2. Centesimal System (French System)
3. Circular System

1. Sexagesimal System: In this system, a right angle is divided into 90 equal parts called *degrees*. Each degree is divided into 60 equal parts called *minutes* and each minute is further, subdivided into 60 equal parts called *seconds*.

Thus, 1 right angle = 90 degrees
1 degree = 60 minutes
1 minute = 60 seconds.

In symbols, a degree, a minute and a second are respectively written as $1°$; $1'$, $1''$. Thus, $40° 15' 20''$ denotes the angle which contains 40 degrees 15 minutes and 20 seconds. The unit of measurement in this system is degree. This system is called sexagesimal because each unit is divided into 60 parts (sexagesimus means sixtieth) so that number 60 comes which marking the divisions.

2. Centesimal System: In this system, a right angle is divided into 100 equal parts called *grades*. Each grade is divided into 100 equal parts called *minutes* and each minute is divided into 100 equal parts called *seconds*.

Thus, 1 right angle = 100 grades
1 grade = 100 minutes
1 minute = 100 seconds

In this system the symbols 1^g, $1^`$, $1^{``}$ stand for a *grade*, a minute and a second respectively. Thus, $20^g\ 12^`\ 85^{``}$ denotes the angle which contains 20 grades 12 minutes and 85 seconds. The unit of measurement here is grade. This system is called centesimal system because the number 100 comes in marking the divisions (centesimus means hundredth).

Note. The reader should observe the difference in notations of minutes and seconds in the Sexagesimal and Centesimal systems.

3. Circular System: In this system, the unit of measurement is radian. It is defined as the angle subtended at the centre of a circle by an Arc that is equal to the radius of the circle.

NOTES

Consider a circle with centre O. Take any point A on it and cut off an Arc AB that has a length, equal to the radius of the circle. Then, $\angle AOB$ is called a radian (see Figure 1.2).

The symbol 1^c denotes a radian.

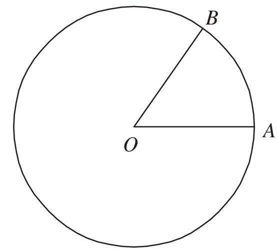


Figure 1.2 *Radian Angle*

It is well known that '*The circumference of a circle bears a constant ratio to its diameter.*'

This constant ratio is denoted by the Greek letter π (pronounced as 'pie').

The value of π correct upto two places of decimal is 3.14 equivalent to $\frac{22}{7}$ and upto six places of decimal is 3.141593, equivalent to $\frac{355}{113}$.

Radian is a Constant Angle

Consider a circle with centre O. Let A be any point on it and AB be an Arc equal to the radius OA (see Figure 1.3). Then,

$$\frac{\angle AOB}{\text{Arc } AB} = \frac{\angle AOA'}{\text{Arc } AA'}$$

i.e.

$$\frac{\angle AOB}{\angle AOA'} = \frac{\text{Arc } AB}{\text{Arc } AA'}$$

$$= \frac{\text{Radius}}{\frac{1}{2}\text{ Circumference}}$$

$$= \frac{\text{Diameter}}{\text{Circumference}} = \frac{1}{\pi}$$

Therefore, $\angle AOB = \dfrac{1}{\pi}\angle AOA'$

$$= \frac{1}{\pi} \times 2 \text{ right angles}$$

$$= \frac{180^\circ}{\pi}\text{, which is constant as } \pi \text{ is a constant.}$$

NOTES

Thus one radian = $\frac{180^\circ}{\pi}$

Hence, π radians = 180° (see Figure 1.3).

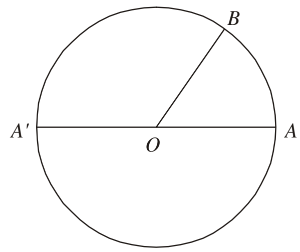


Figure 1.3 *Radian Angle of 180°*

1.2.3 Relations between the Three Systems of Measurements

Since 1 right (rt) angle = 90° and 100^g = 1 right angle

We have 90° = 100^g so that 180° = 200^g

But 180° = π radians

Hence, 180° = 200^g = π^c.

Note**:** Sometimes the superscript '*c*' is omitted while writing the angles, so that angle θ means, an angle of magnitude θ radians. Therefore, we shall be writing π both for an angle as well as number so that angle π stands for π radians and number π stands for ratio of circumference of a circle to its diameter.

Angle subtend by an Arc at the centre of a circle.

To prove that the number of radians in an angle subtended by an Arc of a circle at the centre is equal to $\frac{\textbf{Arc}}{\textbf{Radius}}$

Proof: Consider a circle with centre *O* (see Figure 1.4). Let *A* be any point on it. Let Arc *AB* = radius *OA*. Let *C* be any point on the circle. Then,

$$\angle AOB = 1 \text{ radian}$$

Now, $$\frac{\angle AOC}{\angle AOB} = \frac{\text{Arc } AC}{\text{Arc } AB}$$

$$\Rightarrow \quad \angle AOC = \frac{\text{Arc } AC}{\text{Arc } AB} \times 1 \text{ radian}$$

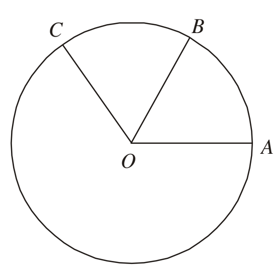


Figure 1.4 *Circle*

NOTES

So the number of radians in $\angle AOC = \dfrac{\text{Arc AC}}{\text{Radius}}$

Note**:** If θ is number of radians in an angle; *l*, the length of Arc of the circle subtending an angle θ at the centre of the circle and *r* the radius of circle, then $\theta = \dfrac{l}{r}$.

Example 1.1: Express in terms of right angles and also in grades, minutes and seconds the following angles:

(i) 30º, *(ii)* 138º 30′, *(iii)* 35º 47′ 15″.

Solution: (*i*) We have $90^\circ = 100^g$,

Also $\quad 90^\circ = 1$ rt. angle

So $\quad 1^\circ = \dfrac{10}{9}^g$

and $\quad 30^\circ = \dfrac{1}{3}$ rt. angle $= 0.33$ rt. angles

So that $\quad 30^\circ = \dfrac{300}{9}^g = 33\dfrac{1}{3}^g$

Now, $\quad 1^g = 100`$

$\Rightarrow \quad \dfrac{1}{3}^g = \dfrac{100`}{3} = 33\dfrac{1`}{3}$

Also, $\quad 1` = 100``$

$\Rightarrow \quad \dfrac{1`}{3} = \dfrac{100``}{3} = 33.3``$

$\Rightarrow \quad 30^\circ = 33^g\, 33`\, 33.3``$.

(*ii*) We have, $\quad 30` = \dfrac{1}{2}^\circ = 0.5^\circ$

$\Rightarrow \quad 138^\circ\, 30` = 138.5^\circ$

Now, $\quad 90^\circ = 1$ rt. angle

$\Rightarrow \quad 1^\circ = \dfrac{1}{90}$ rt. angle

$\Rightarrow \quad 138.5^\circ = \dfrac{138.5}{90}$ rt. angle $= 1.5388888$ rt. angle

Now 1 rt. angle $= 100^g$

$\Rightarrow \quad 1.5388888$ rt. angle $= 153.88888^g$

Also, $\quad .88888^g = 88.888`$

and $\quad .888` = (0.888 \times 100)`` = 88.88``$

$\Rightarrow \quad 138.5 = 153^g\, 88`\, 88.88``$

(*iii*) We have, $\quad 15`` = \dfrac{15`}{60} = \dfrac{1`}{4} = 0.25`$

and $\quad 47`\, 15`` = 47.25` = \dfrac{47.25^\circ}{60} = 0.7875^\circ$.

$\Rightarrow \quad 35^\circ\, 47`\, 15`` = 35.7875^\circ$

NOTES

$$= \frac{35.7875}{90} \text{ rt. angle}$$
$$= 0.3976388 \text{ rt. angle} = 39.76388^g$$

Now, $0.76388^g = 76.388`$

$0.388` = 38.88``$

$\Rightarrow$ $35^\circ\, 47`\, 15`` = 39^g\, 76`\, 38.88``$

Example 1.2: Express in terms of right angles and also in degrees, minutes and seconds the following angles:

(i) 120^g *(ii)* $45^g\, 35`\, 24``$.

Solution: (*i*) We have $1^g = \frac{9^\circ}{10}$.

Also, $100^g = 1$ rt. angle.

So, $120^g = 108^\circ = \frac{6}{5}$ rt. angle

(*ii*) $24`` = 0.24`$

$\Rightarrow$ $35`\, 24`` = 35.24` = 0.3524^g$

Thus $45^g\, 35`\, 24`` = 45.3524^g$

$= 0.453524$ rt. angle

$= (0.453524 \times 90)^\circ = 40.81716^\circ$

Now $.81716^\circ = (0.81716 \times 60)` = 49.0296`$

and $.0296` = (0.0296 \times 60)`` = 1.776``$

Hence, $45^g\, 35`\, 24`` = 40^\circ\, 49`\, 1.776``$

Example 1.3: Convert $5^\circ\, 37`\, 30``$ into radians.

Solution: We have, $30`` = \frac{1`}{2}$

$$\Rightarrow \quad 37`\, 30`` = 37\frac{1`}{2} = \frac{75`}{2} = \left(\frac{75}{2 \times 60}\right)^\circ = \frac{5}{8}^\circ$$

Then $5^\circ\, 37`\, 30`` = 5\frac{5}{8}^\circ = \frac{45}{8}^\circ$.

$$\text{Now} \quad 1^\circ = \frac{\pi^c}{180} \quad \Rightarrow \quad \frac{45^\circ}{8} = \left(\frac{\pi}{180} \times \frac{45}{8}\right)^c = \frac{\pi^c}{32}$$

Example 1.4: Convert $1^g\, 1`$ into radians.

Solution: You have $1` = \frac{1^g}{100}$

$$\Rightarrow \quad 1^g\, 1` = 1\frac{1^g}{100} = \frac{101^g}{100}$$

$$\text{Now} \quad 1^g = \frac{\pi^c}{200}$$

$$\Rightarrow \quad \frac{101^g}{100} = \left(\frac{\pi \times 101}{20000}\right)^c = 0.00505\, \pi^c$$

Example 1.5. If D, G, C are the number of degrees, grades and radians in an angle respectively, prove that:

$$\frac{D}{90} = \frac{G}{100} = \frac{2C}{\pi}.$$

NOTES

Solution: The given angle = D degrees = $\frac{D}{90}$ of a rt. angle.

Also, the given angle = G grades = $\frac{G}{100}$ of a rt. angle.

Further, the given angle = C radians = $\frac{2C}{\pi}$ of a rt. angle.

Hence, $$\frac{D}{90} = \frac{G}{100} = \frac{2C}{\pi}$$

Example 1.6: The number of degrees in a certain angle added to the number of grades in the angle is 152. Find the angle.

Solution: Let x be the number of degrees in the angle.

Then, the number of grades in the angle will be $\frac{10}{9}x$.

So, $$x + \frac{10}{9}x = 152$$

or $$\frac{19}{9}x = 152, \text{ i.e., } x = \frac{152 \times 9}{19} = 72.$$

Example 1.7: The angles of a triangle are in A.P., the number of grades in the least to the number of radians in the greatest is 40 to π, find the angles in degrees.

Solution: Let the angles of a triangle in A.P. be.

$$(a - d)^\circ, a^\circ, (a + d)^\circ$$

Since the sum of angles of a triangle is 180°.

$$\Rightarrow \quad (a - d) + a + (a + d) = 180^\circ$$

$$\Rightarrow \quad a = 60$$

So, the angles are $(60 - d)^\circ, 60^\circ, (60 + d)^\circ$

$$\text{The least angle} = (60 - d)^\circ = \left(\frac{(60 - d) \times 10}{9}\right)^g$$

$$\text{The greatest angle} = (60 + d)^\circ = \left((60 + d) \times \frac{\pi}{180}\right)^c$$

Hence, $$\frac{(60 - d) \times \frac{10}{9}}{(60 + d) \times \frac{\pi}{180}} = \frac{40}{\pi}$$

Hence, $$\frac{(60 - d) \times 200}{60 + d} = 40 \Rightarrow (60 - d) \times 5 = 60 + d$$

$$\Rightarrow \quad 300 - 5d = 60 + d$$

$$\Rightarrow \quad 6d = 240 \Rightarrow d = 40$$

Hence the angles are $(60 - 40)^\circ, 60^\circ, (60 + 40)^\circ$ or $20^\circ, 60^\circ, 100^\circ$.

NOTES

Example 1.8: The angles of a pentagon are in A.P. and greatest is three times the least. Find the angles in grades.

Solution: Let the angles of the pentagon in A.P. be $(a-2d)^\circ$, $(a-d)^\circ$, a°, $(a+d)^\circ$, $(a+2d)^\circ$.

Now, sum of the angles of polygon of n sides is

$$= (2n-4) \text{ right angles}$$

$\Rightarrow$ Sum of angles of pentagon is 6 right angles = 540°

$\Rightarrow$ $(a-2d)+(a-d)+a+(a+d)+(a+2d) = 540^\circ$

$\Rightarrow$ $5a = 540^\circ$ or $a = 108^\circ$

Hence, the angles are

$(108-2d)^\circ$, $(108-d)^\circ$, 108°, $(108+d)$, $(108+2d)^\circ$

The greatest angle = $(108+2d)^\circ$

The least angle = $(108-2d)^\circ$

By hypothesis,

$$(108+2d) = 3(108-2d) \quad \text{or} \quad 108+2d = 324-6d$$

or $$8d = 216 \quad \text{or} \quad d = 27.$$

Hence, the angles are 54°, 81°, 108°, 135°, 162°

or $60^g, 90^g, 120^g, 150^g, 180^g$.

Example 1.9: A cow is tied to a post by a rope. If the cow moves along a circular path always keeping the rope tight and describes 44 ft., when it has traced out 72° at the centre. Find the length of the rope.

Solution: The cow starts from A and describes an Arc of length (see Figure 1.5)

$$l = 44 \text{ ft.} = AB$$

Also $\angle AOB = 72^\circ = \left(72 \times \dfrac{\pi}{180}\right)^c = \left(\dfrac{2\pi}{5}\right)^c$

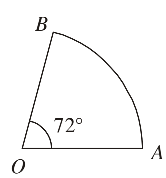


Figure 1.5

If the length of rope be r, then

$$\theta = \frac{l}{r} \Rightarrow \frac{2\pi}{5} = \frac{44}{r}$$

$$\Rightarrow r = \frac{44 \times 5}{2\pi} = \frac{44 \times 5 \times 7}{2 \times 22} = 35 \text{ ft.} \left(\pi = \frac{22}{7}\right)$$

NOTES

Example 1.10: The large hand of a big clock is 3 feet long. How many inches does its extremity move in 10 minutes time?

Solution: The large hand of a clock starts from A (see Figure 1.6) and describes an Arc of length $l = AB$ in 10 minutes.

Now, 60 minutes = 360º

$\Rightarrow$ 10 minutes = 36º

Thus $\angle AOB = 36^\circ = \dfrac{\pi^c}{5}$

Also $OA = 3$ feet $= r$

$$\text{So} \qquad \theta = \frac{l}{r} \quad \Rightarrow \quad \frac{\pi}{5} = \frac{l}{3}$$

$$\Rightarrow \quad l = \frac{3\pi}{5} = \frac{3 \times 22}{5 \times 7} = \frac{66}{35} \text{ feet}$$

$$\Rightarrow \quad l = \frac{66}{35} \times 12 \text{ inches} = 22.63 \text{ inches.}$$

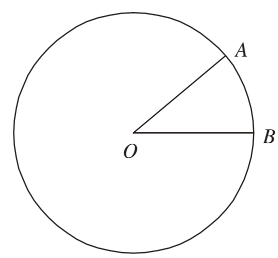


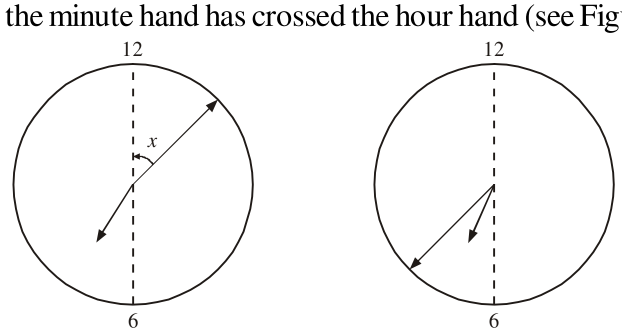
Figure 1.6

Example 1.11: Find the times between 6 O'clock and 7 O'clock when the angle betwen minute hand and hour hand is 29º.

Solution: There are two possibilities:

(i) When the minute hand has not crossed the hour hand.

(ii) When the minute hand has crossed the hour hand (see Figure 1.7).



(i) (ii)

Figure 1.7

NOTES

(*i*) Suppose after the time is 6 O'clock the minute hand has moved through x^o.

Now, when the minute hand moves through 360º, the hour hand moves through 30º.

So, in this case the minute hand has moved through,

$$\left(\frac{1}{12}\times x\right)^o = \left(\frac{x}{12}\right)^o$$

This is with respect to hour hand.

Therefore, $29^o - \left(\frac{x}{12}\right)^o + x^o = 180^o$

$$\Rightarrow \qquad x = \left(\frac{151\times 12}{11}\right)^o$$

$$\Rightarrow \qquad x = \frac{151\times 12}{11}\times\frac{1}{6} \text{ minutes}$$

$$= \frac{302}{11} \text{ minutes} = 27\frac{5}{11} \text{ minutes}$$

$\Rightarrow$ Time is $27\frac{5}{11}$ minutes past 6.

(*ii*) Proceeding as above,

$$180^o + \left(\frac{x}{12}\right)^o + 29 = x$$

$$\Rightarrow \qquad x = (12\times 19)^o = 38 \text{ minutes}$$

$\Rightarrow$ Time is 38 minutes past 6.

CHECK YOUR PROGRESS

1. When is an angle positive or negative?
2. Define radian.

1.3 TRIGONOMETRIC FUNCTIONS

Trigonometry deals with the problem of measurement, solution of triangles and periodic functions. The applications of trigonometry to business cycles and other situations do not specifically involve triangle. They are concerned with the properties and applications of circular or periodic functions.

1.3.1 Periodic Functions

A function is periodic with period $p (p \neq 0)$ if $f(x + p) = f(x)$.

NOTES

The cyclic curve in Figure 1.8 shows a periodic curve with period p.

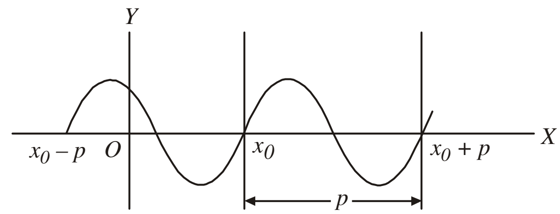


Figure 1.8 *Periodic Curve with Period p*

Radian Measure of an Angle

$\angle LON$ is called a directed angle which is measured by the rotation about its vertex O (see Figure 1.9). The angle is positive if the rotation is anticlockwise and negative if the rotation is clockwise.

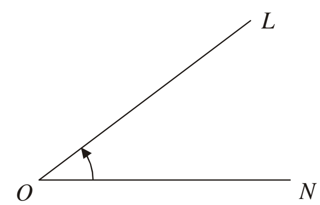


Figure 1.9 *Directed Angle*

The radian measure of an angle expresses degrees in terms of radians. π radians, written π^c, corresponds to 180º.
Similarly, $2\pi^c$ corresponds to 360º. Usually, c is not written to express radian measures (see Figure 1.10).

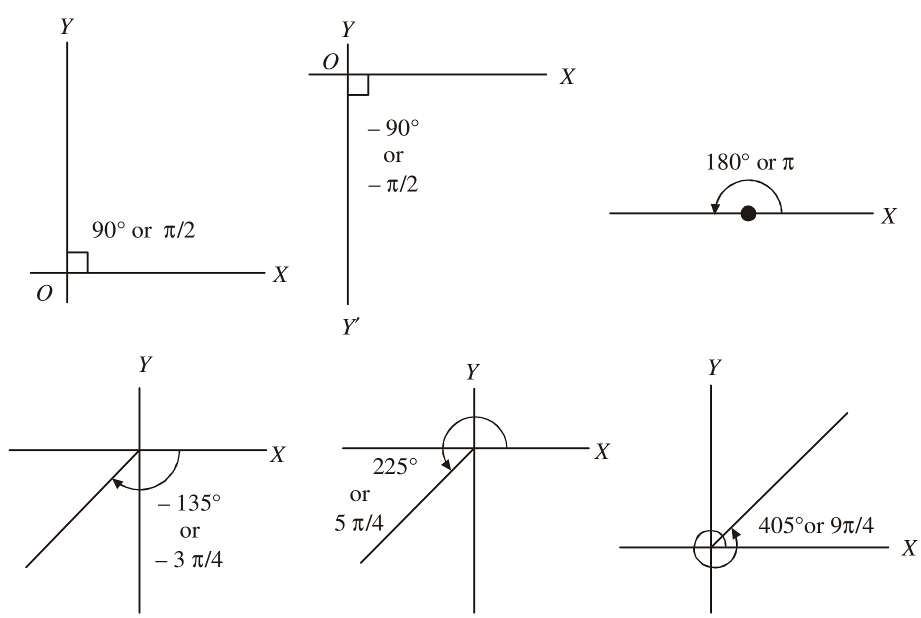


Figure 1.10 *Measures of Angle*

An angle α^o can be converted into radians by multiplying it by $\pi/180$. Thus

$$\alpha^o = \alpha\frac{\pi}{180} \text{ radians}$$

Conversely θ radians can be written as:

$$\theta^c = \alpha\frac{180}{\pi} \text{ degrees}$$

NOTES

Example 1.12: Convert from degree to radian and radian to degree.

(i) 45° to radian (ii) 90° to radian (iii) $\pi/6$ radian to degree

(iv) 60 to radian (v) $\frac{5\pi}{12}$ radian to degree (vi) $\frac{4\pi}{3}$ radian to degree

(vii) $-\frac{5\pi}{16}$ radian to degree (viii) – 30° radian

Solution: (i) $45^\circ = \frac{\pi^c}{4}$ (ii) $\frac{\pi^c}{2} = 90^\circ$ (iii) $\frac{\pi^c}{6} = 30^\circ$

(iv) $60^\circ = \frac{\pi^c}{3}$ (v) $\frac{5\pi^c}{12} = 75^\circ$ (vi) $\frac{4\pi^c}{3} = 240^\circ$

(vii) $\frac{-5\pi^c}{6} = -150^\circ$ (viii) $-30^\circ = -\frac{\pi^c}{6}$.

Measurement of Angles

Two systems of measuring and comparing angles may be considered.

In the sexagesimal system, a right angle obtained by a quarter revolution is divided into 90 equal parts and each part equals one degree, written 1°.

$$1^\circ = 60 \text{ minutes written } 60'$$

$$1' = 60 \text{ seconds written } 60''$$

In the circular measure system, the unit is a radian. A radian is the measure of the angle made at the centre of a circle by an Arc whose length equals the radius of the circle.

The notation used is, 1 radian = 1^c

The circumference of a circle equals $2\pi r$ where r is the radius of the circle. It can be shown that:

$$\frac{1}{2}\pi \text{ radians} = \left(\frac{\pi}{2}\right)^c = 90^\circ$$

Therefore, $$1^c = \left(\frac{180}{\pi}\right) \quad \text{or} \quad 1^\circ = \left(\frac{\pi}{180}\right)^c$$

Since 1^c is calculated by taking $\pi = 3.141593$ and it comes to 57.2958 and thus, $1^c = 57° \; 17' \; 45''$

Normally, approximate value of π is taken as, $\pi = \dfrac{22}{7}$ or 3.142, 180° $= \pi$ radians, 360° = 2π radians

$$\frac{\pi}{10} \text{ radians } = 18°, \; \frac{\pi}{6} \text{ radians} = 30°$$

Area of a circle

The area of a circle with radius r is (see Figure 1.11) πr^2.

The area of a sector AOB subtending an angle θ at the centre is $\dfrac{1}{2} r^2\theta$.

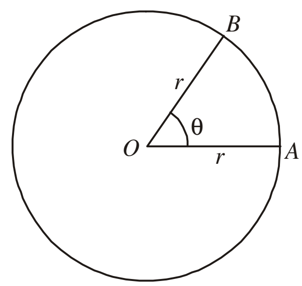


Figure 1.11 *Area of a Circle*

Length of the Arc $AB = r\theta$.

Example 1.13: An Arc $AB = 55$ cm subtends an angle of 150° at the centre of a circle. Find the area of the sector AOB.

Solution: Given $\quad \theta = 150° = 150 \times \dfrac{\pi}{180} = \dfrac{5}{6} \times \dfrac{22}{7} = \left(\dfrac{55}{21}\right)^c$

$$\text{Arc } AB = r\theta = r \times \frac{55}{21} = 55 \qquad \therefore \quad r = 21$$

Therefore, Area of $AOB = \dfrac{1}{2} r^2\theta = \dfrac{1}{2} \times 21 \times 21 \times \dfrac{55}{21} = \dfrac{1155}{2}$ sq. cm.

Area of circle $= \pi r^2 = \dfrac{22}{7} \times 21 \times 21 = 1386$ sq. cm.

1.3.2 Trigonometric Ratios

Take a point $P(x, y)$ on the line OR which makes an angle θ with the x-axis (see Figure 1.12).

In ΔOPM, $OM = x$ and $MP = y$

The hypotenuse $OP = r$

NOTES

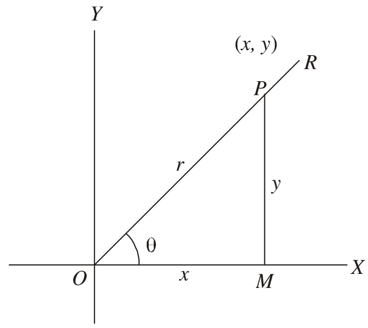


Figure 1.12 *Trigonometric Ratios*

OM is the side adjacent to angle θ.

MP is the side opposite angle θ.

There are six trigonometric ratios:

Sine of θ is $\sin\theta = \frac{\text{Opp. side}}{\text{Hypotenuse}} = \frac{MO}{OP} = \frac{y}{r}$

Cosine of θ is $\cos\theta = \frac{\text{Adjacent side}}{\text{Hypotenuse}} = \frac{OM}{OP} = \frac{x}{r}$

Tangent of θ is $\tan\theta = \frac{\text{Opp. side}}{\text{Adj. side}} = \frac{MP}{OM} = \frac{y}{x}$

Cosecant of θ is $\text{cosec}\,\theta = \frac{1}{\sin\theta} = \frac{r}{y}$

Secant of θ is $\sec\theta = \frac{1}{\cos\theta} = \frac{r}{x}$

Cotangent of θ is $\cot\theta = \frac{1}{\tan\theta} = \frac{x}{y}$

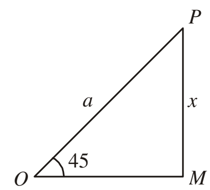
Note: The trigonometric ratios remain the same wherever the point *P* is taken on *OR*.

1.3.3 Values of Trigonometric Functions of Standard Angles

We can find the numerical values of sin 45º, cos 30º, etc., and use them whenever required. Tables are available to find the values of trigonometric functions of all angles but it is useful to remember some standard values.

You can find sin 45º or $\sin\frac{\pi}{4}$

Note that in the right angled triangle *POM*.



NOTES

$$OP^2 = OM^2 + PM^2$$

If, $\angle POM = 45° = \angle MOP, OM = MP = x$

If $OP = a$ then, $a^2 = 2x^2$ or $a = \sqrt{2}x$ or $x = a/\sqrt{2}$

$$\sin 45° = \frac{PM}{OP} = \frac{x}{a} = \frac{a/\sqrt{2}}{a} = \frac{1}{\sqrt{2}}$$

$\therefore$ $\operatorname{cosec} 45° = \sqrt{2}$

Similarly, $\cos 45° = \frac{1}{\sqrt{2}}$, $\sec 45° = \sqrt{2}$

$\tan 45° = 1$, $\cot 45° = 1$

For an angle of 30°, $\sin 30° = \frac{1}{2}$

These results can be put in a tabular form and remembered.

$\sin 0° = 0$	$\cos 0° = 1$	$\tan 0° = 0$
$\sin 30° = 1/2$	$\cos 30° = \sqrt{3}/2$	$\tan 30° = 1/\sqrt{3}$
$\sin 45° = 1/\sqrt{2}$	$\cos 45° = 1/\sqrt{2}$	$\tan 45° = 1$
$\sin 60° = \sqrt{3}/2$	$\cos 60° = 1/2$	$\tan 60° = \sqrt{3}$
$\sin 90° = 1$	$\cos 90° = 0$	$\tan 90° = \infty$

The trigonometric ratios for other angles can be found by suitable methods. Tables for trigonometric ratios are available. It may be remembered that

- $\sin (90° - \theta) = \cos \theta$
- $\cos (90° - \theta) = \sin \theta$
- $\tan (90° - \theta) = \cot \theta$

Also, the following identities can be easily proved:

$$\frac{\sin\theta}{\cos\theta} = \tan\theta, \quad \frac{\cos\theta}{\sin\theta} = \cot\theta \quad \sin^2\theta + \cos^2\theta = 1$$

$$\sec^2\theta - \tan^2\theta = 1 \qquad \operatorname{cosec}^2\theta - \cot^2\theta = 1$$

For example, (i) $2\tan^2 45° + 3\tan^2 60° = 2 \times (1)^2 + 3(\sqrt{3})^2 = 2 + 9 = 11.$

(ii) $\sin^2 30° + 16\cos^2 60° = 4\left(\frac{1}{2}\right)^2 + 16\left(\frac{1}{2}\right)^2 = 1 + 4 = 5.$

(iii) $4\cos^3 30° - 3\cos 30° = 0$ and

$\tan 45° - 2\sin^2 30° - \cos 60° = 0$

The trigonometric ratios are useful in finding the angles and lengths of the sides of a triangle and for a variety of other purposes.

NOTES

1.3.4 Signs of Trigonometric Ratios

As θ increases from 0 to 2π,

$\sin\theta$ rises from 0 to 1 in Quadrant I (Q. I)

falls from 1 to 0 in Quadrant II (Q. II)

falls from 0 to – 1 in Quadrant III (Q. III)

rises from – 1 to 0 in Quadrant IV (Q. IV)

Similarly, $\cos\theta$ falls from 1 to 0 in Q. I

falls from 0 to – 1 in Q. II

rises from – 1 to 0 in Q. III

rises from 0 to 1 in Q. IV

$\tan\theta$ rises from 0 to ∞ in Q. I

rises from $-\infty$ to 0 in Q. II

rises from 0 to ∞ in Q. III

rises from $-\infty$ to 0 in Q. IV

Therefore, as shown in Figure 1.13, all trigonometric ratios are positive in Q. I

- $\sin\theta$ and cosec θ are positive in Q. II
- $\tan\theta$ and $\cot\theta$ are positive in Q. III
- $\cos\theta$ and $\sec\theta$ are positive in Q . IV

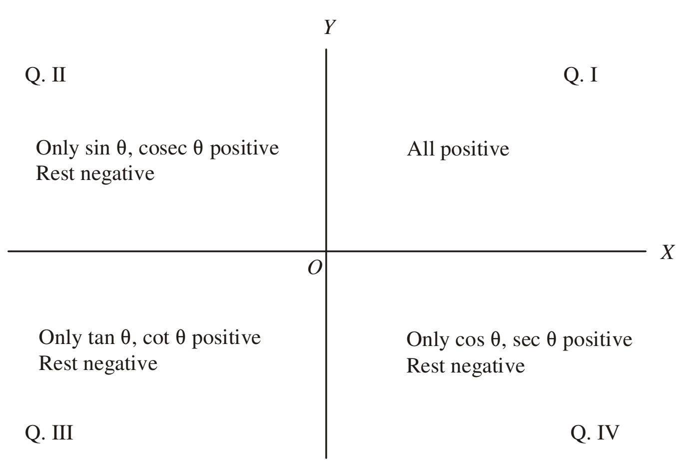


Figure 1.13 *Positive and Negative Trigonometric Ratios*

NOTES

The following results can be verified and used when required:

$$\sin(-\theta) = -\sin\theta$$
$$\cos(-\theta) = \cos\theta$$
$$\tan(-\theta) = -\tan\theta$$

The angle $-\theta$ is in the fourth quadrant in which $\cos(-\theta)$ and $\text{Sec}(-\theta)$ are positive.

$\sin(90-\theta) = \cos\theta$ $\cos(90-\theta) = \sin\theta$ $\tan(90-\theta) = \cot\theta$	The angle $90-\theta$ is in the first quadrant. All ratios are positive.
$\sin(90+\theta) = \cos\theta$ $\cos(90+\theta) = -\sin\theta$ $\tan(90+\theta) = -\cot\theta$	The angle $90+\theta$ is in the second quadrant. Only $\sin(90+\theta)$ is positive.
$\sin(180-\theta) = \sin\theta$ $\cos(180-\theta) = -\cos\theta$ $\tan(180-\theta) = -\tan\theta$	The angle $180-\theta$ is in the second quadrant. Only $\sin(180-\theta)$ is positive.
$\sin(180+\theta) = -\sin\theta$ $\cos(180+\theta) = -\cos\theta$ $\tan(180+\theta) = +\tan\theta$	The angle $180+\theta$ is in the third quadrant. Only $\tan(180+\theta)$ is positive.
$\sin(360-\theta) = -\sin\theta$ $\cos(360-\theta) = \cos\theta$ $\tan(360-\theta) = -\tan\theta$	Same rule as for the angle, $-\theta$.
$\sin(360+\theta) = \sin\theta$ $\cos(360+\theta) = \cos\theta$ $\tan(360+\theta) = \tan\theta$	Same rule as for the angle θ.

Ratios of other angles can be expressed in terms of the above ratios.

$$\sin\left(3\frac{\pi}{2}-\theta\right) = \sin(270-\theta) = \sin(90+180-\theta)$$
$$= \cos(180-\theta) = -\cos\theta$$

or $\quad \sin(270-\theta) = \sin(180+90-\theta)$
$$= -\sin(90-\theta) = -\cos\theta$$

or $\quad \sin(270-\theta) = \sin(270-\theta-360)$
$$= \sin(-90-\theta) = \sin\{-(90+\theta)\} = -\sin(90+\theta)$$
$$= -\cos\theta$$

or $\sin(270-\theta) = -\sin(\theta-270)$

$= -\sin(\theta-270+360) = -\sin(90+\theta) = -\cos\theta$

or $\sin(270-\theta) = -\cos[90+(270-\theta)] = -\cos(360-\theta) = -\cos\theta$

NOTES

$$\sin\frac{3}{2}\pi = \sin(\pi+\pi/2) = -\sin\frac{\pi}{2} = -1$$

$$\cos\frac{5}{4}\pi = -\frac{1}{\sqrt{2}}, \qquad \sin\frac{5}{4}\pi = -\frac{1}{\sqrt{2}}$$

$$\cos\frac{7}{4}\pi = \frac{1}{\sqrt{2}}, \qquad \sin\frac{7}{4}\pi = -\frac{1}{\sqrt{2}}$$

$$\cos\frac{7}{6}\pi = -\frac{\sqrt{3}}{2}, \qquad \sin\frac{7}{6}\pi = -\frac{1}{2}$$

$$\cos\frac{11}{6}\pi = \frac{\sqrt{3}}{2}, \qquad \sin\frac{11}{6}\pi = -\frac{1}{2}$$

$$\cos\frac{2}{3}\pi = -\frac{1}{2} \qquad \sin\frac{5}{3}\pi = -\frac{\sqrt{3}}{2}$$

$$\cos\left(-\frac{3}{4}\pi\right) = -\frac{1}{\sqrt{2}}, \qquad \sin\left(-\frac{4}{3}\pi\right) = \frac{\sqrt{3}}{2}$$

$$\text{If } \sin\theta = -\frac{3}{5}, \text{ then } \cos\theta = \frac{4}{5}, \ \tan\theta = -\frac{3}{4}$$

$$\operatorname{cosec}\theta = -\frac{5}{3}, \ \sec\theta = \frac{5}{4}, \ \cot\theta = -\frac{4}{3}$$

Inclination and Slope of a Line

The inclination of a line corresponds to the angle made by the line with the horizontal axis (see Figure 1.14).

If a line is horizontal or parallel to the x-axis its inclination is zero, i.e., $\theta = 0$. If a line is perpendicular to the x-axis, $\theta = 90°$.

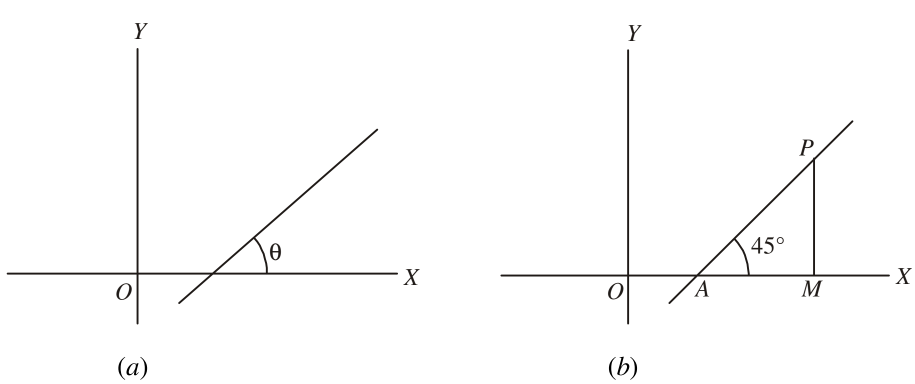


Figure 1.14 *Inclination and Slope of a Line*

If the inclination of a line is θ, the slope of the line is tan θ.

The slope of a line with 45° inclination is tan 45° = 1. Alternatively, since $\angle PAM$ = 45°, $AM = MP$ and tan 45° = MP/AM = 1.

NOTES

For example the slope of a line with inclination (*i*) 30° (*ii*) 60° (*iii*) 0° (*iv*) 90° is

(*i*) $\tan 30^\circ = \dfrac{1}{\sqrt{3}}$ (*ii*) $\tan 60^\circ = \sqrt{3}$ (*iii*) $\tan 0 = 0$ (*iv*) $\tan 90^\circ = \infty$

Note that parallel lines have the same inclination and hence, have equal slopes (see Figure 1.15).

The slope of a curve at a point *P* (see Figure 1.16) is the slope of the tangent drawn at the point *P*.

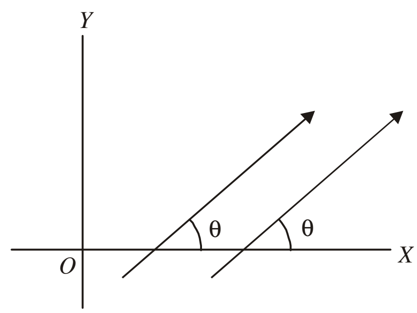


Figure 1.15 *Parallel Lines*

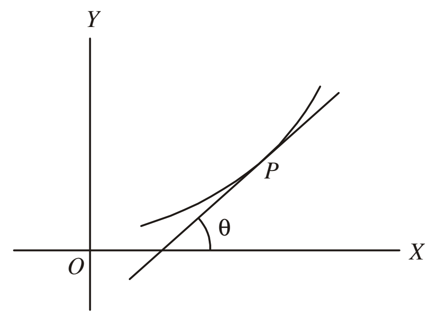


Figure 1.16 *Slope of a Curve at a Point P.*

The intercept of a line on the *y*-axis is the distance from the origin to the point of intersection on the *y*-axis.

OB is the *y* intercept.

Similarly, *OA* is the *x* intercept (see Figure 1.17).

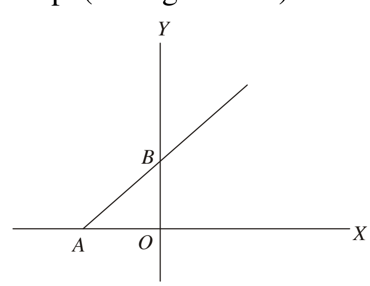


Figure 1.17 *Intercept of a Line*

The intercept may be positive or negative. Zero intercept implies that the line passes through the origin.

***Note*:** We know that the trigonometric ratios;

$$\sin\theta = \frac{MP}{OP},\ \cos\theta = \frac{OM}{OP},\ \tan\theta = \frac{MP}{OM}.$$

NOTES

As OP rotates in a counterclockwise direction, the angle θ changes (see Figure 1.18). While OP remains constant, MP and OM change. Thus, the three ratios are functions of θ. The functions $\sin\theta$, $\cos\theta$, $\tan\theta$ being related to a circle are called circular functions. Being also related to a triangle, they are often referred to as trigonometric functions. They are periodic with the fundamental period 2π.

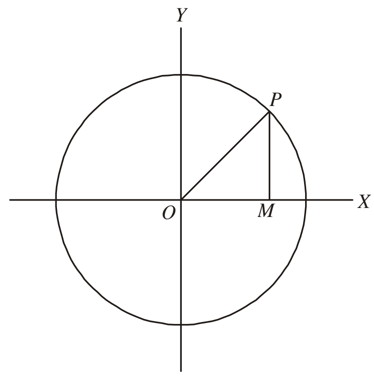


Figure 1.18 *Counterclockwise Rotation*

The graph (see Figure 1.19) of the sine function $f(\theta) = \sin\theta$ repeats itself at intervals of 2π.

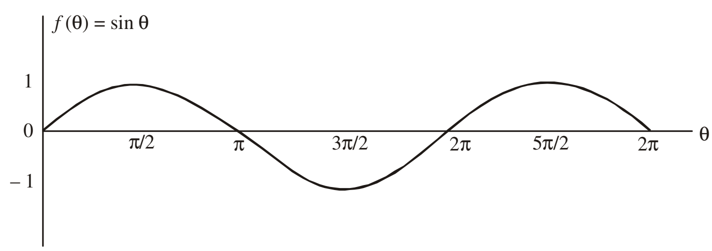


Figure 1.19 *Sin θ Repeating at Intervals 2π*

Since, $\quad \sin(\theta + 2\pi) = \sin(\theta + 360^{\circ}) = \sin\theta$

$$\cos(\theta + 360) = \cos\theta$$

It means that $\sin\theta$ and $\cos\theta$ are periodic functions each with period 2π.

In general, $\quad \sin(\theta + 2n\pi) = \sin\theta$

Similarly, $\quad \cos(\theta + 2n\pi) = \cos\theta$

when n is any integer.

***Note*:** As θ increases from a

(*i*) 0 to $\frac{\pi}{2}$, $\sin\theta$ rises from 0 to 1

(*ii*) $\frac{\pi}{2}$ to π, $\sin\theta$ falls from 1 to 0

(*iii*) π to $3\,\frac{\pi}{2}$, $\sin\theta$ falls from 0 to – 1

NOTES

(*iv*) $\frac{3\pi}{2}$ to 2π, $\sin\theta$ rises from -1 to 0

θ	0	$\pi/4$	$\pi/2$	$3\pi/4$	π	$5\pi/4$	$3\pi/2$	$7\pi/4$	2π
$\sin\theta$	0	$1/\sqrt{2}$	1	$1/\sqrt{2}$	0	$-1/\sqrt{2}$	-1	$-1/\sqrt{2}$	0

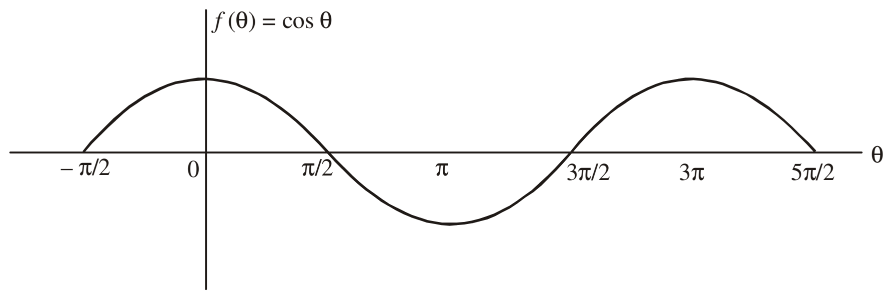


Figure 1.20 *Periodic Graph of cos θ*

The graph (see Figure 1.20) of $f(\theta) = \cos\theta$ is also periodic.

However, $\cos 0 = 1$ and $\cos \pi/2 = 0$.

The amplitude of a sine wave is the absolute value of one half of the difference between the greatest and the least ordinate of the wave. The amplitude in each of the sine and cosine graph is,

$$1/2\{1 - (-1)\} = 1$$

The amplitude of $f(t) = 5\sin t$ is 5. Each ordinate is 5 times the ordinate of $f(t) = \sin t$.

1.3.5 Fundamental Period and Phase

The fundamental period of a periodic function like $f(t) = \sin bt$ or $g(t) = \cos bt$ is given by,

$$T = \frac{2\pi}{|b|}$$

$\sin bt = \sin(bt + 2\pi)$ because the period is 2π.

$$= \sin b\left(t + \frac{2\pi}{b}\right) = f\left(t + \frac{2\pi}{b}\right)$$

The phase of $f(t) = \sin(t - p)$ or $f(t) = \cos(t - p)$ is defined by $|p|$

The phase of $f(t) = \sin\left(t + \frac{1}{2}\pi\right)$ is given by $|p| = \left|-\frac{1}{2}\pi\right| = \frac{\pi}{2}$

For the function,

$$f(t) = 4\cos(2t - \pi/2) = 4\cos 2(t - \pi/4)$$

The amplitude is 4 and phase $\frac{\pi}{4}$.

The fundamental period is $\frac{2\pi}{2} = \pi$.

The amplitude of $2 \sin \frac{1}{2}\pi t$ is 2 and the fundamental period is $\frac{2\pi}{(1/2)\pi} = 4$.

NOTES

Example 1.14: If $f(x) = \sin kx$ is periodic, determine the period for $f(x)$.

Solution: $f(x) = \sin(kx) = \sin(kx + 2\pi) = \sin k(x + 2\pi/k) = f(x + 2\pi/k)$

The period is $2\pi/k$.

This is also true for $f(x) = \cos kx$. The period is $2\pi/k$.

The period for $f(x) = \tan kx$ is π/k.

Example 1.15: Find the period, if any, for $f(x) = \sin x^2$.

Solution: If the period is t, $f(x + t) = \sin (x + t)^2$.

Also $f(x) = \sin x^2 = \sin (x^2 + 2\pi)$

$\therefore \quad (x + t)^2 = x^2 + 2\pi$

$\therefore \quad 2tx + t^2 = 2\pi$ which does not give a fixed value of t.

$f(x) = \sin x^2$

This function is not periodic.

Periodic Functions and Graphs

Graph of $f(\theta) = \sin\theta + 2\cos\theta$ (see Figure 1.21)

$$f(0) = 0 + 2 = 2, \quad f\left(\frac{\pi}{4}\right) = \frac{1}{\sqrt{2}} + 2 \cdot \frac{1}{\sqrt{2}} = 2.1$$

$$f(\pi) = 0 - 2 = -2, \quad f\left(\frac{3}{4}\pi\right) = -\frac{1}{\sqrt{2}} - \frac{2}{\sqrt{2}} = -0.7$$

$$f(2\pi) = 0 + 2 = 2, \quad f\left(\frac{5}{4}\pi\right) = -\frac{1}{\sqrt{2}} - \frac{2}{\sqrt{2}} = -2.1$$

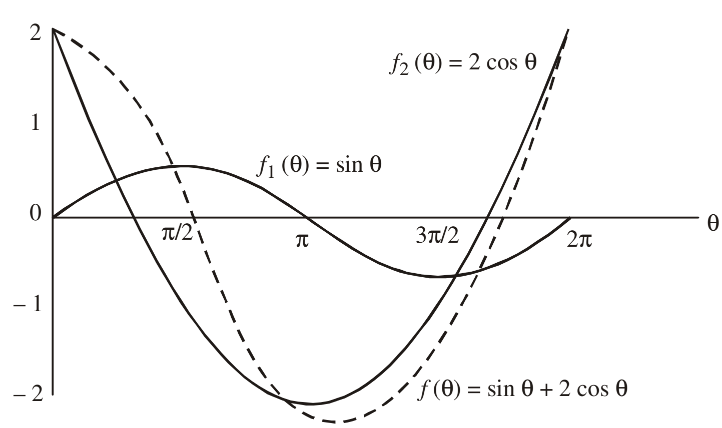


Figure 1.21 *Periodic Functions and Graphs*

Graph as shown in Figure 1.22 of $f(t) = e^{-t} \sin 2t \quad t \geq 0$

$$|f(t)| \leq e^{-t} \text{ because } |\sin 2t| \leq 1 \text{ and } e^{-t} > 0 \text{ for all } t$$

NOTES

i.e., $-e^{-t} \leq f(t) \leq e^{-t}$, i.e., the graph of $f(t)$ lies between the graphs of $f_1(t) = -e^{-t}$ and $f_2(t) = e^{-t}$.

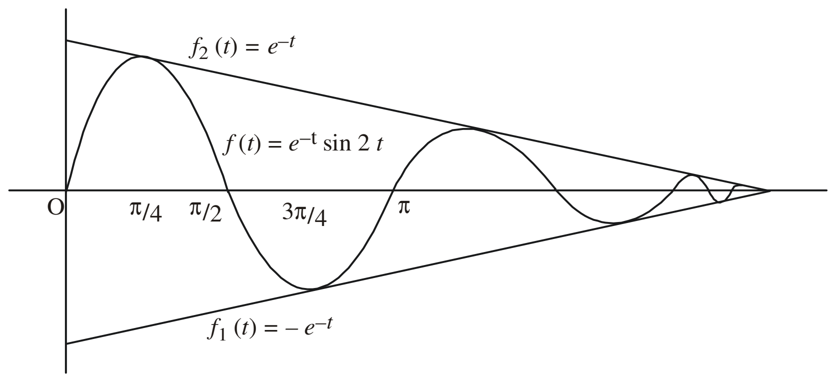


***Figure 1.22** Graph of f(t)*

There are damped oscillations. In the limit, the amplitude is zero.

Trigonometric Ratios

Given any one trigonometric ratio, the other ratios can be found by using the identities provided previously.

Example 1.16: $\cos\theta = -\frac{1}{2}$, find other ratios.

Solution: Note that θ lies in Q. II

To find other ratios, use

$$\sin\theta = \sqrt{1-\cos^2\theta} = \sqrt{1-\frac{1}{4}} = \sqrt{\frac{3}{4}} = \pm\frac{\sqrt{3}}{2}$$

We have taken the value $+\sqrt{3}/2$ since θ lies in Q. II

$$\tan\theta = \sin\theta/\cos\theta = (\sqrt{3}/2)/(-1/2) = -\sqrt{3}$$

$$\text{cosec}\,\theta = 1/\sin\theta = 2/\sqrt{3}$$

$$\sec\theta = 1/\cos\theta = -2$$

$$\cot\theta = 1/\tan\theta = -1/\sqrt{3}$$

Example 1.17: Given: $\tan\theta = \frac{3}{4}$, find other ratios.

Solution: Note that θ lies in Q. III

We have, $\qquad \sec^2\theta = 1 + \tan^2\theta = 1 + 9/16 = \frac{25}{16}$

$$\sec\theta = \pm\frac{5}{4}$$

$$\sec\theta = -\frac{5}{4} \text{ since } \theta \text{ lies in Q. III}$$

$$\cos\theta = -4/5$$

$$\sin\theta = \sqrt{1-\cos^2\theta} = \sqrt{1-\frac{16}{25}} = \sqrt{\frac{9}{25}} = -\frac{3}{5} \text{ (Q. III)}$$

$$\cot\theta = 4/3, \operatorname{cosec}\theta = -5/3.$$

NOTES

Exmaple 1.18: Show $4\cot^2 45° - \sec^2 60° + \sin^2 30° = \frac{1}{4}$.

Solution: $4 \times (1)^2 - (2)^2 + \left(\frac{1}{2}\right)^2 = 4 - 4 + \frac{1}{4} = \frac{1}{4}$

Identities

In the forthcoming exercises, the following identities will be useful.

(*i*) $\sin^2 A + \cos^2 A = 1$, (*ii*) $\sec^2 A = 1 + \tan^2 A$, (*iii*) $\operatorname{cosec}^2 A = 1 + \cot^2 A$

Example 1.19: Prove that: $\sqrt{\frac{1-\sin A}{1+\sin A}} = \sec A - \tan A$

Solution: LHS $= \sqrt{\frac{(1-\sin A)(1-\sin A)}{(1+\sin A)(1-\sin A)}} = \sqrt{\frac{(1-\sin A)^2}{1-\sin^2 A}}$

$$= \sqrt{\frac{(1-\sin A)^2}{\cos^2 A}} = \frac{1-\sin A}{\cos A} = \frac{1}{\cos A} - \frac{\sin A}{\cos A}$$

$$= \sec A - \tan A$$

Example 1.20: Prove that $\frac{\cos A}{1-\tan A} + \frac{\sin A}{1-\cot A} = \sin A + \cos A$.

Solution: Put $\tan A = \sin A/\cos A$ and $\cot A = \cos A/\sin A$

$$\text{LHS} = \frac{\cos A \cdot \cos A}{\cos A - \sin A} + \frac{\sin A \sin A}{\sin A - \cos A} = \frac{\cos^2 A}{\cos A - \sin A} - \frac{\sin^2 A}{\cos A - \sin A}$$

$$= \frac{\cos^2 A - \sin^2 A}{\cos A - \sin A} = \frac{(\cos A + \sin A)(\cos A - \sin A)}{\cos A - \sin A} = \cos A + \sin A.$$

Compound Angles

To find $\sin(A + B)$.

From a point P on OP' draw $PQ \perp OQ'$ (see Figure 1.23).

From Q draw $QK \perp PM$.

NOTES

$$PM = QN + PK$$

$$\frac{PM}{OP} = \frac{QN}{OP} + \frac{PK}{OP}$$

$$= \frac{QN}{OQ} \cdot \frac{OQ}{OP} + \frac{PK}{PQ} \cdot \frac{PQ}{OP}$$

$$= \sin A \cos B + \cos A \sin B$$

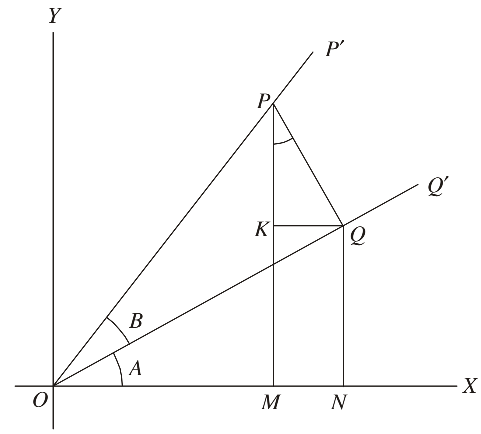


Figure 1.23 *Compound Angles*

Since, $\angle KPQ = \angle A$

$$\sin (A + B) = \frac{PM}{OP} = \sin A \cos B + \cos A \sin B \qquad ...(1.1)$$

Similarly,

$$\cos (A + B) = \frac{OM}{OP} = \frac{ON}{OP} - \frac{MN}{OP} = \frac{ON}{OQ} \cdot \frac{OQ}{OP} - \frac{KQ}{PQ} \cdot \frac{PQ}{OP}$$

$\therefore$ $$\cos (A + B) = \cos A \cos B - \sin A \sin B \qquad ...(1.2)$$

Replacing B by $-B$ in the preceding results,

$$\sin (A - B) = \sin A \cos B - \cos A \sin B \qquad ...(1.3)$$

$$\cos (A - B) = \cos A \cos B + \sin A \sin B \qquad ...(1.4)$$

Also $$\tan (A + B) = \frac{\sin (A + B)}{\cos (A + B)} = \frac{\sin A \cos B + \cos A \sin B}{\cos A \cos B - \sin A \sin B}$$

Dividing the numerator and denominator by $\cos A \cos B$, we have

$$\tan (A + B) = \frac{\frac{\sin A}{\cos A} + \frac{\sin B}{\cos B}}{1 - \frac{\sin A \sin B}{\cos A \cos B}} = \frac{\tan A + \tan B}{1 - \tan A \tan B} \qquad ...(1.5)$$

Similarly, $$\tan (A - B) = \frac{\tan A - \tan B}{1 + \tan A \tan B} \qquad ...(1.6)$$

NOTES

It can be seen by putting $A + B = C$, and $A - B = D$ that

$$\sin C + \sin D = 2 \sin \frac{C+D}{2} \cos \frac{C-D}{2};$$

$$\sin C - \sin D = 2 \cos \frac{C+D}{2} \sin \frac{C-D}{2};$$

$$\cos C + \cos D = 2 \cos \frac{C+D}{2} \cos \frac{C-D}{2};$$

$$\cos C - \cos D = -2 \sin \frac{C+D}{2} \sin \frac{C-D}{2}.$$

We know that, $\sin (A + B) = \sin A \cos B + \cos A \sin B$ ...(1.7)

Put $A = B$, then $\sin (2A) = \sin A \cos A + \cos A \sin A$

We have the result $\sin 2A = 2 \sin A \cos A$ ...(1.8)

On the same lines, using the result for $\cos (A + B)$ we can prove

$$\cos 2A = \cos^2 - \sin^2 A = 2 \cos^2 A - 1 = 1 - 2 \sin^2 A \quad ...(1.9)$$

$$\sin 3A = \sin (2A + A) = \sin 2A \cos A + \cos 2A \sin A$$

$$= 2 \sin A \cos A + (1 - 2 \sin^2 A) \sin A$$

$$= 2 \sin A(1 - \sin^2 A) + (1 - 2 \sin^2 A) \sin A$$

$$= 2 \sin A - 2 \sin^3 A + \sin A - 2 \sin^3 A$$

$\therefore$ $$\sin 3A = 3 \sin A - 4 \sin^3 A \quad ...(1.10)$$

Similarly, $$\cos 3A = \cos (2A + A) = 4 \cos^3 A - 3 \cos A \quad ...(1.11)$$

$$\tan 3A = \tan (2A + A) = \frac{\tan 2A + \tan A}{1 - \tan 2A \tan A}$$

$$= \frac{\dfrac{2 \tan A}{1 - \tan^2 A} + \tan A}{1 - \dfrac{2 \tan A}{1 - \tan^2 A} \tan A} = \frac{2 \tan A + \tan A - \tan^3 A}{1 - \tan^2 A - 2 \tan^2 A}$$

$$= \frac{3 \tan A - \tan^3 A}{1 - 3 \tan^2 A} \quad ...(1.12)$$

In the result $\cos 2A = \cos^2 A - \sin^2 A$, if we replace A by $\frac{A}{2}$ we have

$$\cos A = \cos^2 \frac{A}{2} - \sin^2 \frac{A}{2} = 2 \cos^2 \frac{A}{2} - 1 = 1 - 2 \sin^2 \frac{A}{2}$$

...(1.13)

Similarly, $$\sin A = 2 \sin \frac{A}{2} \cos \frac{A}{2} \quad ...(1.14)$$

$$\tan A = \frac{2\tan A/2}{1-\tan^2 A/2} = \frac{2t}{1-t^2} \text{ where } t = \tan\frac{A}{2} \qquad ...(1.15)$$

The following additional results are useful and may be derived:

NOTES

Since $\cos 2A = 2\cos^2 A - 1 = 1 - 2\sin^2 A$

$\therefore$ $$\cos^2 A = \frac{1+\cos 2A}{2} \quad \text{or} \quad \cos A = \pm\frac{\sqrt{1+\cos 2A}}{2}$$

$$\sin^2 A = \frac{1-\cos 2A}{2} \quad \text{or} \quad \sin A = \pm\frac{\sqrt{1-\cos 2A}}{2}$$

Summary of Some Important Results

I. $$\sin(A \pm B) = \sin A\cos B \pm \cos A\sin B$$

$$\cos(A \pm B) = \cos A\cos B \mp \sin A\sin B$$

$$\sin(A+B+C) = \sin A\cos B\cos C + \cos A\sin B\cos C$$
$$+\cos A\cos B\sin C - \sin A\sin B\sin C$$

$\cos(A+B+C) = \cos A\cos B\cos C - \cos A\sin B\sin C - \sin A\cos B$
$\sin C$

$-\sin A\sin B\cos C$

$$\tan(A+B) = \frac{\tan A+\tan B}{1-\tan A\tan B}, \quad \tan(A-B) = \frac{\tan A-\tan B}{1+\tan A\tan B}$$

$$\tan(A+B+C) = \frac{\tan A+\tan B+\tan C-\tan A\tan B\tan C}{1-\tan B\tan C-\tan C\tan A-\tan A\tan B}$$

$$\cot(A+B) = \frac{\cot A\cot B-1}{\cot B+\cot A} \qquad \cot(A-B) = \frac{\cot A\cot B+1}{\cot B-\cot A}$$

II. $$\sin 2A = 2\sin A\cos A = \frac{2\tan A}{1+\tan^2 A} \qquad \sin A = \frac{2\tan A/2}{1+\tan^2 A/2}$$

$$\cos 2A = \left\{\begin{array}{l} =\cos^2 A-\sin^2 A \\ =1-2\sin^2 A \\ =2\cos^2 A-1 \end{array}\right\} = \frac{1-\tan^2 A}{1+\tan^2 A}; \quad \cos A = \frac{1-\tan^2 A/2}{1+\tan^2 A/2}$$

$$\tan 2A = \frac{2\tan A}{1-\tan^2 A} = \frac{1-\cos^2 A}{1+\cos^2 A}; \qquad \tan A = \frac{2\tan A/2}{1-\tan^2 A/2}$$

$$1-\cos 2A = 2\sin^2 A$$

$$1+\cos 2A = 2\cos^2 A$$

$$\sin 3A = 3\sin A - 4\sin^3 A$$

$$\cos 3A = 4\cos^3 A - 3\cos A$$

$$\tan 3A = \frac{3\tan A - \tan^3 A}{1 - 3\tan^2 A}$$

III. Conversion of product into sum:

NOTES

$$\sin A \cos B = \frac{1}{2}\{\sin(A+B) + \sin(A-B)\}$$

$$\cos A \sin B = \frac{1}{2}\{\sin(A+B) - \sin(A-B)\}$$

$$\cos A \cos B = \frac{1}{2}\{\cos(A+B) + \cos(A-B)\}$$

$$\sin A \sin B = -\frac{1}{2}\{\cos(A+B) - \cos(A-B)\}$$

IV. Conversion of sum into product

$$\sin C + \sin D = 2\sin\frac{C+D}{2}\cos\frac{C-D}{2}$$

$$\sin C - \sin D = 2\cos\frac{C+D}{2}\sin\frac{C-D}{2}$$

$$\cos C + \cos D = 2\cos\frac{C+D}{2}\cos\frac{C-D}{2}$$

$$\cos C - \cos D = -2\sin\frac{C+D}{2}\sin\frac{C-D}{2}$$

V. $$\cos A/2 = \pm\sqrt{\frac{1+\cos A}{2}} \qquad \text{...(1.16)}$$

$$\sin A/2 = \pm\sqrt{\frac{1-\cos A}{2}} \qquad \text{...(1.17)}$$

$$\tan A/2 = \pm\sqrt{\frac{1-\cos A}{1+\cos A}} \qquad \text{...(1.18)}$$

Also, if we write $\tan A/2 = t$, we have

$$\sin A = \frac{2t}{1+t^2}, \quad \cos A = \frac{1-t^2}{1+t^2}, \quad \tan A = \frac{2t}{1-t^2} \qquad \text{...(1.19)}$$

$$\tan\frac{A}{2} = \frac{\sin A/2}{\cos A/2} = \frac{\sin A/2 \cdot 2\cos A/2}{\cos A/2 \cdot 2\cos A/2}$$

$$= \frac{\sin A}{2\cos^2 A/2} = \frac{\sin A}{1+\cos A}.$$

NOTES

Example 1.21: Prove: (i) $\tan^2 A \operatorname{cosec} A = \dfrac{\sin A}{1-\sin^2 A}$

$$(ii) \quad \frac{\sec^2 A - 1}{\sin^2 A} = \frac{1}{\cos^2 A}$$

Solution: (i) $\left(\dfrac{\sin A}{\cos A}\right)^2 \dfrac{1}{\sin A} = \dfrac{\sin^2 A}{\cos^2 A} \cdot \dfrac{1}{\sin A} = \dfrac{\sin A}{\cos^2 A} = \dfrac{\sin A}{1-\sin^2 A}$

$$(ii) \ \frac{\sec^2 A - 1}{\sin^2 A} = \frac{\tan^2 A}{\sin^2 A} = \frac{\sin^2 A}{\cos^2 A} \cdot \frac{1}{\sin^2 A} = \frac{1}{\cos^2 A}$$

Example 1.22: Given $q \cos \theta = p \sin \theta$, find the value of $\dfrac{p \cos \theta + q \sin \theta}{p \cos \theta - q \sin \theta}$.

Solution: Since $q \cos \theta = p \sin \theta \cos \theta \ \dfrac{p}{q} \sin \theta$ and hence, $p \cos \theta = \dfrac{p^2}{q} \sin \theta$, the expression becomes

$$\frac{\frac{p^2}{q} \sin \theta + q \sin \theta}{\frac{p^2}{q} \sin \theta - q \sin \theta} = \frac{\left(\frac{p^2}{q} + q\right) \sin \theta}{\left(\frac{p^2}{q} - q\right) \sin \theta} = \frac{p^2 + q^2}{p^2 - q^2}.$$

Example 1.23: Prove that $\sin (A + B) \sin (A - B) = \sin^2 A - \sin^2 B$.

Solution: LHS $= (\sin A \cos B + \cos A \sin B)(\sin A \cos B - \cos A \sin B)$

$$= \sin^2 A \cos^2 B - \cos^2 A \sin^2 B$$

$$= \sin^2 A \,(1 - \sin^2 B) - (1 - \sin^2 A) \sin^2 B$$

$$= \sin^2 A - \sin^2 A \sin^2 B - \sin^2 B + \sin^2 A \sin^2 B = \sin^2 A - \sin^2 B.$$

Example 1.24: Prove that $\dfrac{\cos 9^\circ + \sin 9^\circ}{\cos 9^\circ - \sin 9^\circ} = \tan 54^\circ$.

Solution: Divide Numerator and Denominator by $\cos 9^\circ$

$$\text{LHS} = \frac{1 + \tan 9^\circ}{1 - \tan 9^\circ} = \frac{\tan 45^\circ + \tan 9^\circ}{1 - \tan 45^\circ \tan 9^\circ} \qquad (\because \ \tan 45^\circ = 1)$$

$$= \tan (45^\circ + 9^\circ) = \tan 54^\circ$$

Example 1.25: Prove that $8 \sin \dfrac{\pi}{14} \sin \dfrac{3\pi}{14} \sin \dfrac{5\pi}{14} = 1$.

Solution: LHS $= 8 \cos\left(\dfrac{\pi}{2} - \dfrac{\pi}{14}\right) \cos\left(\dfrac{\pi}{2} - \dfrac{3\pi}{14}\right) \cos\left(\dfrac{\pi}{2} - \dfrac{5\pi}{14}\right)$

$$= 8 \cos \frac{3\pi}{7} \cos \frac{2\pi}{7} \cos \frac{\pi}{7} = \frac{8}{2 \sin \frac{\pi}{7}} \cdot 2 \sin \frac{\pi}{7} \cos \frac{\pi}{7} \cos \frac{2\pi}{7} \cos \frac{3\pi}{7}$$

$$[2 \sin A \cos A = \sin 2A]$$

$$= \frac{8}{2\sin\pi/7}\sin\frac{2\pi}{7}\cos\frac{2\pi}{7}\cos\frac{3\pi}{7} = \frac{8}{4\sin\frac{\pi}{7}}\sin\frac{4\pi}{7}\cos\frac{3\pi}{7}$$

$$= \frac{8}{8\sin\pi/7}\left(\sin\pi + \sin\frac{\pi}{7}\right) = \frac{1}{\sin\pi/7}(0+\sin\pi/7) = 1.$$

NOTES

Example 1.26: If $A + B + C = \pi$, show that:

$$\cos 2A + \cos 2B + \cos 2C = -1 - 4\cos A\cos B\cos C$$

Solution: LHS $= (2\cos^2 A - 1) + 2\cos(B + C)\cos(B - C)$

Since $B + C = \pi - A$, $\cos(B + C) = \cos(\pi - A) = -\cos A$

$$\begin{aligned}\text{LHS} &= -1 + 2\cos^2 A - 2\cos A\cos(B - C)\\ &= -1 + 2\cos A(\cos A - \cos(B - C))\\ &= -1 - 2\cos A(-\cos(B + C) - \cos(B - C))\\ &= -1 - 2\cos A\, 2\cos B\cos C\\ &= -1 - 4\cos A\cos B\cos C.\end{aligned}$$

Example 1.27: If $\cos\theta + \sin\theta = \sqrt{2}\cos\theta$, show $\cos\theta - \sin\theta = \sqrt{2}\sin\theta$.

Solution: Dividing by $\cos\theta$, we have

$$1 + \tan\theta = \sqrt{2} \text{ or } \tan\theta = \sqrt{2} - 1 < 1, \textit{i.e.}, \sin\theta < \cos\theta$$

Squaring, $\cos\theta + \sin\theta = \sqrt{2}\cos\theta$

$$\cos^2\theta + \sin^2\theta + 2\cos\theta\sin\theta = 2\cos^2\theta$$

$\therefore$ $\cos^2\theta - \sin^2\theta - 2\cos\theta\sin\theta = 0$

$\therefore$ $(\cos\theta - \sin\theta)^2 = 2\sin^2\theta$

$\therefore$ $\cos\theta - \sin\theta = \pm\sqrt{2}\sin\theta$

$\therefore$ $\cos\theta - \sin\theta = \sqrt{2}\sin\theta$

($-\sqrt{2}\sin\theta$ not possible because $\tan\theta < 1$)

Example 1.28: Given $a = \sin A + \sin B$, $b = \cos A + \cos B$, show that:

$$\tan\frac{A - B}{2} = \sqrt{\frac{4 - a^2 - b^2}{a^2 + b^2}}.$$

Solution: We have:

$$\begin{aligned}a^2 + b^2 &= (\sin A + \sin B)^2 + (\cos A + \cos B)^2\\ &= \sin^2 A + 2\sin A\sin B + \sin^2 B + \cos^2 A + 2\cos A\cos B +\\ &\quad \cos^2 B\end{aligned}$$

$$= 1 + 2\,(\sin A \sin B + \cos A \cos B) + 1 = 2 + 2\cos(A - B)$$

$$\frac{4 - a^2 - b^2}{a^2 + b^2} = \frac{4}{a^2 + b^2} - 1 = \frac{4}{2(1 + \cos(A - B))} - 1$$

NOTES

$$= \frac{1 - \cos(A - B)}{1 + \cos(A - B)} = \frac{2\sin^2 \frac{A - B}{2}}{2\cos^2 \frac{A - B}{2}} = \tan^2 \frac{A - B}{2}$$

$$\therefore \quad \sqrt{\frac{4 - a^2 - b^2}{a^2 + b^2}} = \tan \frac{A - B}{2}$$

Example 1.29: Show $\left(\dfrac{\cos A + \cos B}{\sin A - \sin B}\right)^n + \left(\dfrac{\sin A + \sin B}{\cos A - \cos B}\right)^n = 2\cot^n \dfrac{A - B}{2}$ or 0.

Solution: LHS $= \left(\dfrac{2\cos\frac{A+B}{2}\cos\frac{A-B}{2}}{2\cos\frac{A+B}{2}\sin\frac{A-B}{2}}\right)^n + \left(\dfrac{2\cos\frac{A+B}{2}\cos\frac{A-B}{2}}{-2\sin\frac{A+B}{2}\sin\frac{A-B}{2}}\right)^n$

$= 2\cot^n \dfrac{A - B}{2}$ if n is even

$= 0$ if n is odd.

Check Your Progress

3. State the formula for the area of a circle.
4. What is the formula for the area of a sector of a circle?
5. What is the measure of inclination in the following cases:
   (i) A line is either horizontal or parallel to the x-axis?
   (ii) A line is perpendicular to the x-axis?
6. What is the fundamental period of a periodic function like $f(t) = \sin bt$ or $g(t) = \cos bt$?

1.4 IDENTITIES AND SIGNS

Let a revolving line OP start from OX in the anticlockwise direction and trace out an angle XOP. From P draw $PM \perp OX$. Produce OX, if necessary. (see Figure 1.24). Let $\angle XOP = \theta$.

NOTES

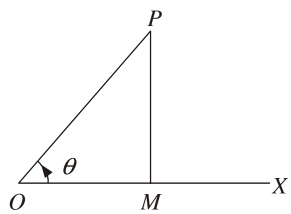


Figure 1.24 *Identities and Signs*

Then, (1) $\frac{MP}{OP}$ is called sine of angle θ and is written as sin θ.

(2) $\frac{OM}{OP}$ is called *cosine* of angle θ and is written as cos θ.

(3) $\frac{MP}{OM}$ is called *tangent* of angle θ and is written as tan θ.

(4) $\frac{OM}{MP}$ is called *cotangent* of angle θ and is written as cot θ.

(5) $\frac{OP}{OM}$ is called *secant* of angle θ and is written as sec θ.

(6) $\frac{OP}{MP}$ is called *cosecant* of angle θ and is written as cosec θ.

These ratios are called Trigonometrical Ratios of the angle θ.

***Notes*:**

1. It follows from the definition that,

$$\sec\theta = \frac{1}{\cos\theta}, \quad \operatorname{cosec}\theta = \frac{1}{\sin\theta}, \quad \cot\theta = \frac{1}{\tan\theta},$$

$$\tan\theta = \frac{\sin\theta}{\cos\theta}, \quad \cot\theta = \frac{\cos\theta}{\sin\theta}.$$

2. Trigonometrical ratios are same for the same angle. For, let P' be any point on the revolving line OP. Draw $P'M' \perp OX$. Then triangles OPM and $OP'M'$ are similar, so $\frac{MP}{OP} = \frac{M'P'}{OP'}$, i.e., each of these ratios is sin θ.

   Therefore, whatever be the triangle of reference (i.e., ΔOPM or $\Delta OP'M'$) might be, we find that sin θ remains the same for a particular angle θ.

   It can be similarly shown that no trigonometrical ratio depends on the size of triangle of reference.

3. $(\sin\theta)^n$ is written as $\sin^n\theta$, where n is positive. Similar notation holds good for other trigonometrical ratios.

4. $\sin^{-1}\theta$ denotes that angle whose sine is θ. Note that $\sin^{-1}\theta$ does not stand for $\frac{1}{\sin\theta}$. Similar notation holds good for other trigonometrical ratios.

NOTES

For any Angle θ,

1. $\sin^2\theta + \cos^2\theta = 1$
2. $\sec^2\theta = 1 + \tan^2\theta$
3. $\text{cosec}^2\theta = 1 + \cot^2\theta$

Proof: Let the revolving line *OP* start from *OX* and trace out an angle θ in the anticlockwise direction. From *P* draw $PM \perp OX$. Produce *OX*, if necessary. Then, $\angle XOP = \theta$.

(1) $\sin\theta = \dfrac{MP}{OP}$, $\cos\theta = \dfrac{OM}{OP}$

Then, $\sin^2\theta + \cos^2\theta = \dfrac{(MP)^2 + (OM)^2}{(OP)^2} = \dfrac{(OP)^2}{(OP)^2} = 1.$

(2) $\sec\theta = \dfrac{OP}{OM}$, $\tan\theta = \dfrac{MP}{OM}$

$$\text{Then, } 1 + \tan^2\theta = 1 + \frac{(MP)^2}{(OM)^2} = \frac{(OM)^2 + (MP)^2}{(OM)^2}$$

$$= \frac{(OP)^2}{(OM)^2} = \left(\frac{OP}{OM}\right)^2 = (\sec\theta)^2 = \sec^2\theta$$

(3) $\cot\theta = \dfrac{OM}{MP}$, $\text{cosec}\,\theta = \dfrac{OP}{MP}$.

$$\text{Then, } 1 + \cot^2\theta = 1 + \left(\frac{OM}{MP}\right)^2 = \frac{(MP)^2 + (OM)^2}{(MP)^2}$$

$$= \frac{(OP)^2}{(MP)^2} = \left(\frac{OP}{MP}\right)^2 = (\text{cosec}\,\theta)^2 = \text{cosec}^2\theta.$$

1.4.1 Signs of Trigonometric Ratios

Consider four lines *OX, OX′*, *OY*, *OY′* at right angles to each other (Figure 1.25). Let a revolving line *OP* start from *OX* in the anticlockwise direction. From *P* draw $PM \perp OX$ or *OX′*. We have the following convention of signs regarding the sides of Δ*OPM*.

1. *OM* is positive, if it is along *OX*.
2. *OM* is negative, if it is along *OX′*.
3. *MP* is negative, if it is along *OY′*.
4. *MP* is positive, if it is along *OY*.
5. *OP* is regarded always positive.

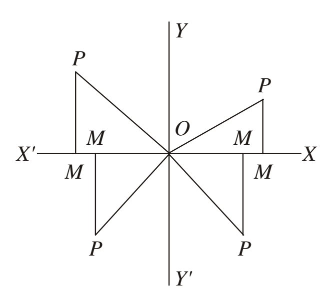


Figure 1.25 *Signs of Trigonometric Ratios*

NOTES

First quadrant. If the revolving line OP is in the first quadrant, then all the sides of the triangle OPM are positive. Therefore, all the trigonometric ratios are positive in the first quadrant.

Second quadrant. If the revolving line OP is in the second quadrant, then OM is negative and the other two sides of ΔOPM are positive. Therefore, ratios involving OM will be negative. So, cosine, secant, tangent, cotangent of an angle in the second quadrant are negative while sine and cosecant of anlge in the second quadrant are positive.

Third quadrant. If the revolving line is in the third quadrant, then sides OM and MP both are negative. Since OP is always positive, therefore, ratios involving each one of OM and MP alone will be negative. So, sine, cosine, cosecant and secant of an angle in the third quadrant are negative. Since tangent or cotangent of any angle involve both OM and MP, therefore, these will be positive. So, tangent and cotangent of an angle in the third quadrant are positive.

Fourth quadrant. If the revolving line OP is in the fourth quadrant, then MP is negative and the other two sides of ΔOPM are positive. Therefore, ratios involving MP will be negative and others positive. So, sine, cosecant, tangent and cotangent of an angle in the fourth quadrant are negative while cosine and secant of an angle in the fourth quadrant are positive.

Limits to the Value of Trigonometric Ratios

We know that $\sin^2\theta + \cos^2\theta = 1$ for any angle θ. $\sin^2\theta$ and $\cos^2\theta$ being perfect squares, will be positive. Again neither of them can be greater than 1 because then the other will have to be negative.

Thus, $\sin^2\theta \le 1$, $\cos^2\theta \le 1$.

$\Rightarrow$ $\sin\theta$ and $\cos\theta$ cannot be numerically greater than 1.

Similarly, $\operatorname{cosec}\theta = \dfrac{1}{\sin\theta}$ and $\sec\theta = \dfrac{1}{\cos\theta}$ cannot be numerically less than 1.

There is no restriction on $\tan\theta$ and $\cot\theta$. They can have any value.

Example 1.30: Prove that $\sin^6\theta + \cos^6\theta = 1 - 3\sin^2\theta\cos^2\theta$.

Solution: Here

$$\begin{aligned}
\text{LHS} &= \sin^6\theta + \cos^6\theta \\
&= (\sin^2\theta)^3 + (\cos^2\theta)^3 \\
&= (\sin^2\theta + \cos^2\theta)(\sin^4\theta - \sin^2\theta\cos^2\theta + \cos^4\theta) \\
&= 1\ .\ (\sin^4\theta - \sin^2\theta\cos^2\theta + \cos^4\theta) \\
&= [(\sin^2\theta + \cos^2\theta)^2 - 3\sin^2\theta\cos^2\theta] \\
&= 1 - 3\sin^2\theta\cos^2\theta = \text{RHS}.
\end{aligned}$$

Example 1.31: Prove that $\sqrt{\dfrac{1+\cos\theta}{1-\cos\theta}} = \operatorname{cosec}\theta + \cot\theta$. Provided $\cos\theta \ne 1$.

NOTES

Solution: LHS = $\sqrt{\dfrac{1+\cos\theta}{1-\cos\theta}} = \sqrt{\dfrac{(1+\cos\theta)(1+\cos\theta)}{(1-\cos\theta)(1+\cos\theta)}} = -\dfrac{1+\cos\theta}{\sqrt{1-\cos^2\theta}}$

$$= \frac{1+\cos\theta}{\sin\theta} = \frac{1}{\sin\theta} + \frac{\cos\theta}{\sin\theta} = \text{cosec }\theta + \cot\theta.$$

Example 1.32: Prove that $(1+\cot\theta-\text{cosec }\theta)(1+\tan\theta+\sec\theta)=2$.

Solution: LHS $= (1+\cot\theta-\text{cosec }\theta)(1+\tan\theta+\sec\theta)$

$$= \left(1+\frac{\cos\theta}{\sin\theta}-\frac{1}{\sin\theta}\right)\left(1+\frac{\sin\theta}{\cos\theta}+\frac{1}{\cos\theta}\right)$$

$$\text{LHS} = \frac{(\sin\theta+\cos\theta-1)(\cos\theta+\sin\theta+1)}{\sin\theta\cos\theta}$$

$$= \frac{(\sin\theta+\cos\theta)^2-1}{\sin\theta\cos\theta}$$

$$= \frac{\sin^2\theta+\cos^2\theta+2\sin\theta\cos\theta-1}{\sin\theta\cos\theta}$$

$$= \frac{1+2\sin\theta\cos\theta-1}{\sin\theta\cos\theta} = \frac{2\sin\theta\cos\theta}{\sin\theta\cos\theta} = 2 = \text{RHS}.$$

Example 1.33: Prove that $\dfrac{\tan\theta}{1-\cot\theta}+\dfrac{\cot\theta}{1-\tan\theta} = 1 + \text{cosec }\theta\sec\theta$, if $\cot\theta \neq 1, 0$ and $\tan\theta \neq 1, 0$.

Solution: LHS $= \dfrac{\tan\theta}{1-\cot\theta}+\dfrac{\cot\theta}{1-\tan\theta}$

$$= \frac{\tan\theta}{1-\dfrac{1}{\tan\theta}} + \frac{\dfrac{1}{\tan\theta}}{1-\tan\theta}$$

$$= \frac{\tan^2\theta}{\tan\theta-1} + \frac{1}{\tan\theta(1-\tan\theta)}$$

$$= \frac{\tan^2\theta}{\tan\theta-1} - \frac{1}{\tan\theta(\tan\theta-1)}$$

$$= \frac{\tan^3\theta-1}{\tan\theta(\tan\theta-1)}$$

$$= \frac{(\tan\theta-1)(\tan^2\theta+\tan\theta+1)}{\tan\theta(\tan\theta-1)}$$

$$= \frac{\tan^2\theta+\tan\theta+1}{\tan\theta} \quad \text{since } \tan\theta\neq 1$$

$$= \frac{\sec^2\theta+\tan\theta}{\tan\theta}$$

$$= \frac{\sec^2\theta}{\tan\theta}+1 = \sec\theta\,\text{cosec }\theta + 1 = \text{RHS}.$$

NOTES

Example 1.34: Which of the six trigonometrical ratios are positive for (i) 960º (ii) – 560º?

Solution: (*i*) 960º = 720º + 240º.

Therefore, the revolving line starting from *OX* will make two complete revolutions in the anticlockwise direction and further trace out an angle of 240º in the same direction. Thus, it will be in the third quadrant. So, the tangent and cotangent are positive and rest of trigonometrical ratios will be negative.

(*ii*) – 560º = – 360º – 200º.

Therefore, the revolving line after making one complete revolution in the clockwise direciton, will trace out further an angle of 200º in the same direction. Thus, it will be in the second quadrant. So, only sine and cosecant are positive.

Example 1.35: In what quadrants can θ lie if sec θ = $\frac{-7}{6}$?

Solution: As sec θ is negative in second and third quadrants, θ can lie in second or third quadrant only.

Example 1.36: If $\sin\theta = \frac{-12}{13}$, determine other trigonometrical ratios of θ.

Solution: $\cos^2\theta = 1 - \sin^2\theta$

$$= 1 - \frac{144}{169} = \frac{169 - 144}{169} = \frac{25}{169}$$

$$\Rightarrow \quad \cos\theta = \pm\frac{5}{13}$$

So $$\tan\theta = \frac{\sin\theta}{\cos\theta} = \mp\frac{12}{5}$$

$$\operatorname{cosec}\theta = \frac{-13}{12},\ \sec\theta = \pm\frac{13}{5},\ \cot\theta = \mp\frac{5}{12}.$$

Example 1.37: Express all the trigonometrical ratios of θ in terms of the sin θ.

Solution: Let sin θ = *k*.

Then, $\cos^2\theta = 1 - \sin^2\theta = 1 - k^2$

$$\Rightarrow \quad \cos\theta = \pm\sqrt{1-k^2} \quad \pm\sqrt{1-\sin^2\theta}$$

$$\tan\theta = \frac{\sin\theta}{\cos\theta} = \pm\frac{k}{\sqrt{1-k^2}} = \pm\frac{\sin\theta}{\sqrt{1-\sin^2\theta}}$$

$$\cot\theta = \frac{\cos\theta}{\sin\theta} = \pm\frac{\sqrt{1-k^2}}{k} = \pm\frac{\sqrt{1-\sin^2\theta}}{\sin\theta}$$

$$\sec\theta = \frac{1}{\cos\theta} = \pm\frac{1}{\sqrt{1-k^2}} = \pm\frac{1}{\sqrt{1-\sin^2\theta}}$$

$$\operatorname{cosec}\theta = \frac{1}{\sin\theta} = \frac{1}{k} = \frac{1}{\sin\theta}$$

Example 1.38: Prove that $\sin\theta = a + \frac{1}{a}$ is impossible, if a is real.

NOTES

Solution: $\sin\theta = a + \frac{1}{a} \Rightarrow \sin\theta = \frac{a^2+1}{a}$

$$\Rightarrow a^2 - a\sin\theta + 1 = 0$$

$$\Rightarrow a = \frac{\sin\theta \pm \sqrt{\sin^2\theta - 4}}{2}$$

For a to be real, the expression under the radical sign, must be positive or zero.

i.e., $\sin^2\theta - 4 \geq 0$

or $\sin^2\theta \geq 4 \Rightarrow \sin\theta$ is numerically greater than or equal to 2 which is impossible.

Thus, if a is real, $\sin\theta = a + \frac{1}{a}$ is impossible.

Example 1.39: Prove that:

$$\frac{1}{\operatorname{cosec}\theta + \cot\theta} - \frac{1}{\sin\theta} = \frac{1}{\sin\theta} - \frac{1}{\operatorname{cosec}\theta - \cot\theta}$$

Solution: $\text{LHS} = \frac{1}{\operatorname{cosec}\theta + \cot\theta} - \frac{1}{\sin\theta}$

$$= \frac{\sin\theta}{1+\cos\theta} - \frac{1}{\sin\theta}$$

$$= \frac{\sin^2\theta - (1+\cos\theta)}{(1+\cos\theta)\sin\theta}$$

$$= \frac{-(1-\sin^2\theta) - \cos\theta}{(1+\cos\theta)\sin\theta}$$

$$= \frac{-\cos^2\theta - \cos\theta}{(1+\cos\theta)\sin\theta}$$

$$= \frac{-\cos\theta(1+\cos\theta)}{(1+\cos\theta)\sin\theta} = -\cot\theta$$

$$\text{RHS} = \frac{1}{\sin\theta} - \frac{1}{\operatorname{cosec}\theta - \cot\theta}$$

$$= \frac{1}{\sin\theta} - \frac{\sin\theta}{1-\cos\theta}$$

$$= \frac{1-\cos\theta-\sin^2\theta}{\sin\theta(1-\cos\theta)}$$

$$= \frac{\cos^2\theta - \cos\theta}{\sin\theta(1-\cos\theta)}$$

$$= \frac{-\cos\theta(1-\cos\theta)}{\sin\theta(1-\cos\theta)} = -\cot\theta$$

Therefore, LHS = RHS.

Example 1.40: Prove that:

$\sin\theta(1+\tan\theta)+\cos\theta\,(1+\cot\theta)=\sec\theta+\operatorname{cosec}\theta.$

Solution: LHS $=\sin\theta\,(1+\tan\theta)+\cos\theta\,(1+\cot\theta)$

NOTES

$$=\sin\theta\left(1+\frac{\sin\theta}{\cos\theta}\right)+\cos\theta\left(1+\frac{\cos\theta}{\sin\theta}\right)$$

$$=\sin\theta+\frac{\sin^2\theta}{\cos\theta}+\cos\theta+\frac{\cos^2\theta}{\sin\theta}$$

$$=\frac{\sin^2\theta\cos\theta+\sin^3\theta+\cos^2\theta\sin\theta+\cos^3\theta}{\sin\theta\cos\theta}$$

$$=\frac{\sin^2\theta(\sin\theta+\cos\theta)+\cos^2\theta\,(\sin\theta+\cos\theta)}{\sin\theta\cos\theta}$$

$$=\frac{(\sin^2\theta+\cos^2\theta)\,(\sin\theta+\cos\theta)}{\sin\theta\cos\theta}$$

$$=\frac{\sin\theta+\cos\theta}{\sin\theta\cos\theta}$$

$$=\frac{1}{\cos\theta}+\frac{1}{\sin\theta}=\sec\theta+\operatorname{cosec}\theta=\text{RHS}.$$

Example 1.41: State giving the reason whether the following equation is possible:

2 sin² θ – 3 cos θ – 6 = 0

Solution: $2\sin^2\theta-3\cos\theta-6=0$

$\Rightarrow\quad 2(1-\cos^2\theta)-3\cos\theta-6=0$

$\Rightarrow\quad -2\cos^2\theta-3\cos\theta-4=0$

$\Rightarrow\quad 2\cos^2\theta+3\cos\theta+4=0$

$$\Rightarrow\quad \cos\theta=\frac{-3\pm\sqrt{9-32}}{4}=\frac{-3\pm\sqrt{-23}}{4}$$

$\Rightarrow\quad \cos\theta$ is imaginary, hence this equation is not possible

Example 1.42: Prove that:

$$\frac{1-\sin\theta}{1+\sec\theta}-\frac{1+\sin\theta}{1-\sec\theta}=2\cos\theta\,(\cot\theta+\operatorname{cosec}^2\theta)$$

Solution: LHS $=\dfrac{(1-\sin\theta)\cos\theta}{1+\cos\theta}-\dfrac{(1+\sin\theta)\cos\theta}{\cos\theta-1}$

$$=\cos\theta\left[\frac{(1-\sin\theta)}{(1+\cos\theta)}+\frac{(1+\sin\theta)}{(1-\cos\theta)}\right]$$

NOTES

$$= \cos\theta \left[\frac{\begin{array}{c}(1-\sin\theta)(1-\cos\theta)\\ +(1+\sin\theta)(1+\cos\theta)\end{array}}{1-\cos^2\theta} \right]$$

$$= \cos\theta \left[\frac{\begin{array}{c}1-\sin\theta-\cos\theta+\sin\theta\cos\theta\\ +1+\sin\theta+\cos\theta+\sin\theta\cos\theta\end{array}}{\sin^2\theta} \right]$$

$$= \cos\theta \left[\frac{2+2\sin\theta\cos\theta}{\sin^2\theta} \right]$$

$$= 2\cos\theta\,[\operatorname{cosec}^2\theta + \cot\theta] = \text{RHS}.$$

Example 1.43: If $\tan x = \dfrac{\sin\theta - \cos\theta}{\sin\theta + \cos\theta}$ where θ and x are both positive and acute angles, prove that:

$$\sin x = \frac{1}{\sqrt{2}}(\sin\theta - \cos\theta)$$

Solution: Since $\tan x = \dfrac{\sin\theta - \cos\theta}{\sin\theta + \cos\theta}$ $\quad 1+\tan^2 x = 1 + \dfrac{\sin^2\theta + \cos^2\theta - 2\sin\theta\cos\theta}{\sin^2\theta + \cos^2\theta + 2\sin\theta\cos\theta}$

$$= 1 + \frac{(1-2\sin\theta\cos\theta)}{(1+2\sin\theta\cos\theta)} = \frac{2}{1+2\sin\theta\cos\theta}$$

Therefore, $\sec^2 x = \dfrac{2}{1+2\sin\theta\cos\theta}$

$$\Rightarrow \quad \cos^2 x = \frac{1+2\sin\theta\cos\theta}{2}$$

$$\Rightarrow \quad 1-\cos^2 x = \frac{2-(1+2\sin\theta\cos\theta)}{2}$$

$$= \frac{1-2\sin\theta\cos\theta}{2} = \frac{(\sin\theta-\cos\theta)^2}{2}$$

$$\Rightarrow \quad \sin^2 x = \frac{(\sin\theta-\cos\theta)^2}{2}$$

$$\Rightarrow \quad \sin x = \pm\frac{(\sin\theta-\cos\theta)}{\sqrt{2}}.$$

Since θ is acute and $\tan x \geq 0$, $\sin\theta \geq \cos\theta$

$\Rightarrow \quad \sin\theta - \cos\theta \geq 0$

Also x is acute $\Rightarrow \sin x \geq 0$

$$\Rightarrow \quad \sin x = +\frac{(\sin\theta-\cos\theta)}{\sqrt{2}}.$$

NOTES

Example 1.44: Express

$$(\sin\theta - 3)(\sin\theta - 1)(\sin\theta + 1)(\sin\theta + 3) + 16$$

as a perfect square and examine if there is any suitable value of θ for which the above expression can be removed.

Solution: Now, $(\sin\theta - 3)(\sin\theta - 1)(\sin\theta + 1)(\sin\theta + 3) + 16$

$$= (\sin^2\theta - 1)(\sin^2\theta - 9) + 16$$
$$= \sin^4\theta - 10\sin^2\theta + 25$$
$$= (\sin^2\theta - 5)^2.$$

This is 0 only when $\sin^2\theta - 5 = 0$, i.e., only when $\sin^2\theta = 5$

Which is not possible as the maximum value of $\sin^2\theta$ is 1.

Thus, there is no value of θ for which the given expression can vanish.

Example 1.45: Show that:

$$\frac{\tan\theta}{\sec\theta - 1} + \frac{\tan\theta}{\sec\theta + 1} = 2\operatorname{cosec}\theta.$$

Solution: LHS= $\dfrac{\tan\theta}{\sec\theta - 1} + \dfrac{\tan\theta}{\sec\theta + 1}$

$$= \tan\theta\left[\frac{1}{\sec\theta - 1} + \frac{1}{\sec\theta + 1}\right]$$
$$= \tan\theta\left[\frac{2\sec\theta}{\sec^2\theta - 1}\right]$$
$$= \tan\theta\left[\frac{2\sec\theta}{\tan^2\theta}\right]$$
$$= \frac{2\sec\theta}{\tan\theta} = \frac{2}{\sin\theta} = 2\operatorname{cosec}\theta = \text{RHS}.$$

CHECK YOUR PROGRESS

7. Determine the quadrant in which θ must lie if cot θ is positive and cosec θ is negative.
8. If $\tan\theta = \dfrac{4}{5}$, find the value of,
$$\frac{2\sin\theta + 3\cos\theta}{4\cos\theta + 3\sin\theta}$$
9. Find the value in terms of p and q of,
$$\frac{p\cos\theta + q\sin\theta}{p\cos\theta - q\sin\theta} \text{ where } \cot\theta = \frac{p}{q}.$$

1.5 TRIGONOMETRIC RATIOS OF ANGLES

1.5.1 Standard Angles

NOTES

Angles of 45°, 60°, 30°

***Angle of* 45º.** Let the revolving line OR starting from OX trace out an angle of 45º (see Figure 1.26) in the anticlockwise direction. Take any point P on OR. From P draw $PM \perp OX$.

Then in ΔOMP,

$$\angle MOP = 45^\circ,$$
$$\angle OMP = 90^\circ$$
$$\Rightarrow \quad \angle OPM = 45^\circ.$$

Then, $OM = MP = a$ (say)

Also, $(OP)^2 = (OM)^2 + (MP)^2$
$= a^2 + a^2 = 2a^2$

$\Rightarrow \quad OP = \sqrt{2a}$

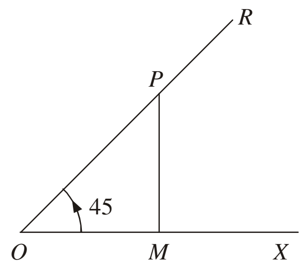


Figure 1.26 *45° Angle*

Now, $\sin 45^\circ = \dfrac{MP}{OP} = \dfrac{a}{\sqrt{2a}} = \dfrac{1}{\sqrt{2}}$

$$\cos 45^\circ = \frac{OM}{OP} = \frac{a}{\sqrt{2a}} = \frac{1}{\sqrt{2}}$$

$$\tan 45^\circ = \frac{MP}{OM} = \frac{a}{a} = 1$$

$$\cot 45^\circ = \frac{1}{\tan 45^\circ} = 1$$

$$\text{cosec } 45^\circ = \frac{1}{\sin 45^\circ} = \sqrt{2}$$

$$\sec 45^\circ = \frac{1}{\cos 45^\circ} = \sqrt{2}$$

***Angle of* 60º.** Let the revolving line OR starting from OX trace out an angle of 60º in the anticlockwise direction (see Figure 1.27). Take any point P on OR. From P draw $PM \perp OX$. Take a point M' on OX such that $MM' = OM = a$ (say)

Then $\angle MOP = 60^\circ$,

$\angle OPM = 30^\circ$.

The two Δs OMP and $MM'P$ are congruent.

NOTES

So, $OP = M'P$

and $\angle MOP = \angle OM'P$.

So that $\Delta OM'P$ is an equilateral triangle.

Then $OP = 2OM = 2a$

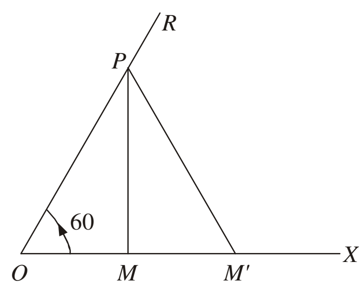


Figure 1.27 *60° Angle*

and $\quad (MP)^2 = (OP)^2 - (OM)^2 = 4a^2 - a^2 = 3a^2$

$$MP = \sqrt{3}a$$

Hence, $\quad \sin 60° = \dfrac{MP}{OP} = \dfrac{\sqrt{3}a}{2a} = \dfrac{\sqrt{3}}{2}$

$$\cos 60° = \frac{OM}{OP} = \frac{a}{2a} = \frac{1}{2}$$

$$\tan 60° = \frac{MP}{OP} = \sqrt{3}$$

$$\cot 60° = \frac{1}{\sqrt{3}},\ \operatorname{cosec} 60° = \frac{2}{\sqrt{3}},\ \sec 60° = 2$$

***Angle of* 30°.** Let the revolving line trace out an angle $XOR = 30°$ in the anticlockwise direction (see Figure 1.28). Take any point P on OR. From P draw $PM \perp OX$. Produce PM *to* P' making $MP' = PM$.

Then $\angle MOP = 30°$,

$\angle OPM = 60°$.

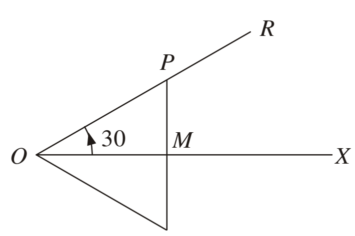


Figure 1.28 *Congruent Triangles When Two Sides are Equal*

The two Δs OPM and $OP'M$ are congruent as two sides are equal and the included angles are equal.

Then $\angle OPM = \angle OP'M = 60°$

so that $\Delta OPP'$ is equilateral.

Let $\quad MP = MP' = a$ (say)

Then, $\quad OP = 2a \quad$ and $\quad OM = \sqrt{3}a$

NOTES

Hence, $\sin 30^\circ = \dfrac{MP}{OP} = \dfrac{a}{2a} = \dfrac{1}{2}$

$\cos 30^\circ = \dfrac{OM}{OP} = \dfrac{\sqrt{3}a}{2a} = \dfrac{\sqrt{3}}{2}$

$\tan 30^\circ = \dfrac{MP}{OM} = \dfrac{a}{\sqrt{3}a} = \dfrac{1}{\sqrt{3}}$

$\cot 30^\circ = \sqrt{3}$, $\sec 30^\circ = \dfrac{2}{\sqrt{3}}$, $\operatorname{cosec} 30^\circ = 2$

What is Infinity?

Consider the fraction a/n, where a is a fixed positie number and n is any positive number. As we give smaller values to n, the fraction a/n becomes larger and larger, and so a/n can be made as large as we like by giving sufficiently small values to n. This fact is expressed by saying 'a/n approaches infinity as n approaches zero' and is written in symbols as,

$$\lim_{n\to 0} \frac{a}{n} = \infty$$

If a is a negative quantity then, as n approaches zero, a/n is said to approach $-\infty$.

***Angle of* 0º.** Let the revolving line OR starting from OX in the anticlockwise direction trace out a very small angle $XOR = \theta$. Take any point P on OR. Let $PM \perp OX$ (see Figure 1.29).

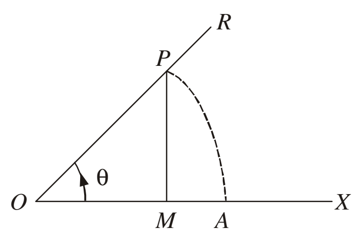


Figure 1.29 *0° Angle*

Draw an Arc of circle with centre O and radius OP, cutting OX at A. Then as θ tends to zero, OM tends to OA and MP tends to zero.

Then, $\sin 0^\circ = \lim_{\theta\to 0^\circ} \sin\theta = \lim_{\theta\to 0^\circ} \dfrac{\text{MP}}{\text{OP}} = \dfrac{0}{OA} = 0.$

$\cos 0^\circ = \lim_{\theta\to 0^\circ} \cos\theta = \lim_{\theta\to 0^\circ} \dfrac{\text{OM}}{\text{OP}} = \dfrac{OA}{OA} = 1.$

$\tan 0^\circ = \lim_{\theta\to 0^\circ} \tan\theta = \lim_{\theta\to 0^\circ} \dfrac{MP}{OM} = \dfrac{0}{OA} = 0.$

$\cot 0^\circ = \lim_{\theta\to 0^\circ} \cot\theta = \lim_{\theta\to 0^\circ} \dfrac{OM}{MP} = \infty.$

$\sec 0^\circ = \lim_{\theta\to 0^\circ} \sec\theta = \lim_{\theta\to 0^\circ} \dfrac{OP}{OM} = \dfrac{OA}{OA} = 1.$

$\operatorname{cosec} 0^\circ = \lim_{\theta\to 0^\circ} \operatorname{cosec}\theta = \lim_{\theta\to 0^\circ} \dfrac{OP}{MP} = \infty.$

NOTES

***Angle of* 90º.** Let the revolving line OR starting from OX trace out an angle θ in the anticlockwise direction, very nearly equal to 90º. Take any point P on OR (*see* Figure 1.30).

Let $PM \perp OX$ and $OY \perp OX$. With O as centre and OP as radius, draw an Arc of a circle cutting OY at B and OX at A.

Then as θ tends to 90º, OP approaches OB, PM approaches OB and OM tends to zero.

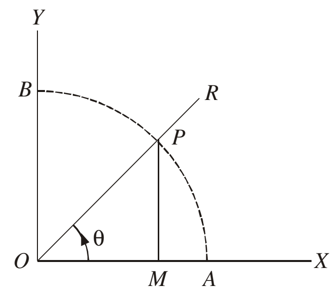


Figure 1.30 *90° Angle*

Then,
$$\sin 90^\circ = \lim_{\theta \to 90^\circ} \sin\theta = \lim_{\theta \to 90^\circ} \frac{MP}{OP} = \frac{OB}{OB} = 1.$$
$$\cos 90^\circ = \lim_{\theta \to 90^\circ} \cos\theta = \lim_{\theta \to 90^\circ} \frac{OM}{OP} = \frac{0}{OB} = 0.$$
$$\tan 90^\circ = \lim_{\theta \to 90^\circ} \tan\theta = \lim_{\theta \to 90^\circ} \frac{MP}{OM} = \infty.$$
$$\cot 90^\circ = \lim_{\theta \to 90^\circ} \cot\theta = \lim_{\theta \to 90^\circ} \frac{OM}{MP} = \frac{0}{OB} = 0.$$
$$\sec 90^\circ = \lim_{\theta \to 90^\circ} \sec\theta = \lim_{\theta \to 90^\circ} \frac{OP}{OM} = \infty.$$
$$\operatorname{cosec} 90^\circ = \lim_{\theta \to 90^\circ} \operatorname{cosec}\theta = \lim_{\theta \to 90^\circ} \frac{OP}{MP} = \frac{OB}{OB} = 1.$$

The results of this section have been summarized in Table 1.1. You are advised to make yourself familiar with it.

Table 1.1 *Standard Angles and their Corresponding Values*

Angle	0º	30º	45º	60º	90º
sine	0	$\frac{1}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{\sqrt{3}}{2}$	1
cosine	1	$\frac{\sqrt{3}}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{1}{2}$	0
tangent	0	1	1	$\sqrt{3}$	∞
cotangent	∞	$\sqrt{3}$	1	$\frac{1}{\sqrt{3}}$	0
cosecant	∞	2	$\sqrt{2}$	$\frac{2}{\sqrt{3}}$	1
secant	1	$\frac{2}{\sqrt{3}}$	$\sqrt{2}$	2	∞

NOTES

Example 1.46: Find the value of cot 60° tan 30° + $\sec^2 45°$.

Solution: $\cot 60° \tan 30° + \sec^2 45° = \frac{1}{\sqrt{3}}\frac{1}{\sqrt{3}} + 2 = \frac{1}{3} + 2 = \frac{7}{3}$

Example 1.47: Find x, if $\tan^2 45° - \cos^2 60° = x \sin 45° \cos 45° \tan 60°$.

Solution: $\quad \text{LHS} = 1 - \frac{1}{4} = \frac{3}{4}$

$$\text{RHS} = x\frac{1}{\sqrt{2}}\frac{1}{\sqrt{2}}\sqrt{3} = \frac{\sqrt{3}x}{2}$$

Then $\quad \frac{3}{4} = \frac{\sqrt{3}x}{2} \quad \Rightarrow \quad x = \frac{\sqrt{3}}{2}$

Example 1.48: If $\theta = 30°$, verify that:

(*i*) $\sin 3\theta = 3 \sin \theta - 4 \sin^3 \theta$

(*ii*) $\cos 3\theta = 4 \cos^3 \theta - 3 \cos \theta$.

Solution: (*i*) $\sin 3\theta = \sin 90° = 1$

and $\quad 3 \sin \theta - 4 \sin^3 \theta = 3 \sin 30° - 4 \sin^3 30°$

$$= \frac{3}{2} - \frac{4}{8} = \frac{3}{2} - \frac{1}{2} = 1.$$

This proves that $\sin 3\theta = 3 \sin \theta - 4 \sin^3 \theta$

(*ii*) when $\theta = 30°$, $\cos 3\theta = \cos 90° = 0$

and $\quad 4 \cos^3 \theta - 3 \cos \theta = 4 \cos^3 30° - 3 \cos 30°$

$$= 4\frac{3\sqrt{3}}{8} - 3\frac{\sqrt{3}}{2} = 0.$$

Example 1.49: Find a solution of the following equation:

$$\cot \theta + \tan \theta = 2 \operatorname{cosec} \theta.$$

Solution: $\cot \theta + \tan \theta = \frac{1}{\tan \theta} + \tan \theta = \frac{1 + \tan^2 \theta}{\tan \theta} = \frac{\sec^2 \theta}{\tan \theta}$.

Then, $\quad \frac{\sec^2 \theta}{\tan \theta} = 2 \operatorname{cosec} \theta$

$\Rightarrow \quad \sec^2 \theta = 2 \operatorname{cosec} \theta \tan \theta = 2 \sec \theta$

$\Rightarrow \quad \sec \theta (\sec \theta - 2) = 0$

$\Rightarrow \quad \sec \theta = 0 \quad \text{or} \quad 2.$

As $\quad \sec \theta = 0$ is impossible.

we get $\quad \sec \theta = 2 \quad \Rightarrow \quad$ one value of $\theta = \pi/3$.

Thus, a solution of given equation is $\theta = \pi/3$.

Example 1.50: Prove that cot 30°, cot 45°, and cot 60° are in G.P.

Solution: Now, $\cot 30° = \sqrt{3} = a$ (say)

$\cot 45° = 1 = b$ (say)

$\cot 60° = \frac{1}{\sqrt{3}} = c$ (say)

Then, $b^2 = 1 = ac$.

Thus, a, b, c are in G.P.

i.e., cot 30º, cot 45º and cot 60º are in G.P.

Example 1.51: Find the value of θ from the following equation:

NOTES

$$\cot^2\theta - (1+\sqrt{3})\cot\theta + \sqrt{3} = 0, \text{ for } 0 < \theta < \frac{\pi}{2}.$$

Solution: Now, $\cot\theta = \dfrac{(1+\sqrt{3}) \pm \sqrt{1+3+2\sqrt{3}-4\sqrt{3}}}{2}$

$$= \frac{(1+\sqrt{3}) \pm (1-\sqrt{3})}{2} = 1,\ \sqrt{3}$$

$\cot\theta = 1 \quad \Rightarrow \quad \theta = \dfrac{\pi}{4}$

$\cot\theta = \sqrt{3} \quad \Rightarrow \quad \theta = \dfrac{\pi}{3}$

Therefore, there are two values of θ, namely $\frac{\pi}{4}$ and $\frac{\pi}{3}$ which satisfy the given equation.

Example 1.52: ABC is a right-angled triangle in which the angle C is a right angle and BC = $\frac{1}{2}$ AB. The line AD bisecting the angle A meets BC at D. Obtain the value of tan 15º from Figure 1.31.

Solution: Let $AB = 2x$.

Then, $BC = x$.

Let $\angle CAB = 2\theta$

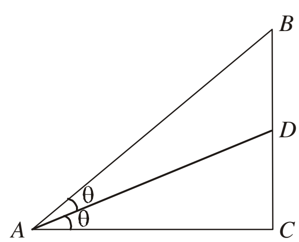


Figure 1.31

Then, $\sin 2\theta = \dfrac{BC}{AB} = \dfrac{1}{2}$

$\Rightarrow 2\theta = 30^\circ \quad \Rightarrow \quad \theta = 15^\circ.$

Now, $\dfrac{AC}{AB} = \dfrac{CD}{BD}$

and $\quad AC = \sqrt{4x^2 - x^2} = \sqrt{3}x$

$\Rightarrow \quad \dfrac{\sqrt{3}x}{2x} = \dfrac{CD}{BD}$

$\Rightarrow \quad CD = \dfrac{\sqrt{3}}{2} BD$

NOTES

Now, $BD + CD = x$

$\Rightarrow \quad \frac{2}{\sqrt{3}} CD + CD = x$

$\Rightarrow \quad CD = \frac{x\sqrt{3}}{(2+\sqrt{3})}$

Now, from ΔACD

$$\tan 15^\circ = \frac{CD}{AC}$$

$$= \frac{x\sqrt{3}}{(2+\sqrt{3})} \cdot \frac{1}{\sqrt{3}x} = \frac{1}{2+\sqrt{3}} = \sqrt{3} - 2$$

1.5.2 Trigonometric Ratios of Allied Angles

The figures in this section are drawn so that the revolving line lies in the first quadrant. The figures when revolving line lies in second, third or fourth quadrant can be similarly drawn. The same proofs hold good in other cases too.

Angle **(– θ):** Let the revolving line OR starting from OX, move in anticlockwise direction and trace out an angle $XOR = \theta$. Let another revolving line OR' starting from OX, move in clockwise direction and trace out an angle $XOR' = -\theta$ (see Figure 1.32).

Take a point P on OR.

Draw $PM \perp OX$ and produce it to meet OR' at P'.

Δs OPM and $OP'M$ are congruent.

By convention of signs,

$$MP = -MP'$$

$$OP = OP'$$

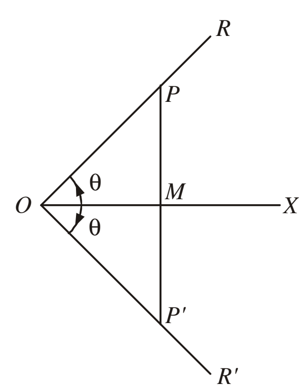


Figure 1.32 *Allied Angles θ*

So, $\sin(-\theta) = \frac{MP'}{OP'} = \frac{-MP}{OP}$

$$= -\sin\theta$$

$$\cos(-\theta) = \frac{OM}{OP'} = \frac{OM}{OP}$$

$$= \cos \theta.$$

$$\tan(-\theta) = \frac{MP'}{OM} = \frac{-MP}{OM}$$

$$= -\tan \theta.$$

$$\cot(-\theta) = \frac{OM}{MP'} = \frac{-OM}{MP} = -\cot \theta.$$

$$\sec(-\theta) = \frac{OP'}{OM} = \frac{OP}{OM} = \sec \theta.$$

$$\operatorname{cosec}(-\theta) = \frac{OP'}{MP'} = \frac{-OP}{MP} = -\operatorname{cosec} \theta.$$

Thus, when θ is changed to $-\theta$, $\cos \theta$ and $\sec \theta$ remain unaltered both in magnitude and sign. All other trigonometrical ratios remain unaltered in magnitude but the sign is changed.

***Angle* (90 – θ):** Let a revolving line OR starting from OX, move in anticlockwise direction and trace an angle $XOR = \theta$ (see Figure 1.33).

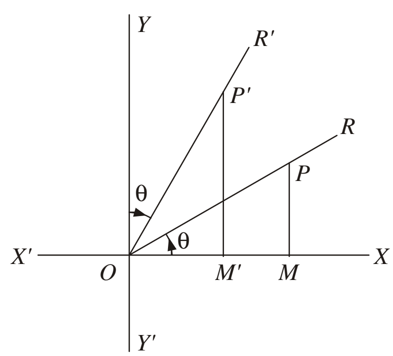


Figure 1.33 *Angle 90 – θ*

Let OR' be another revolving line starting from OX in anticlockwise direction, trace out an angle of 90º and then revolve back through angle θ. Thus, OR' has traced an angle

$$XOR' = 90 - \theta.$$

Take P and P' on OR and OR' respectively such that $OP = OP'$. From P and P' draw $PM \perp OX$ and $P'M' \perp OX$.

Then, Δs OPM and $P'OM'$ are congruent.

Thus, we have $OM' = MP$

$$OM = M'P'.$$

Then, from $\Delta OM'P'$

$$\sin(90-\theta) = \frac{M'P'}{OP'} = \frac{OM}{OP} = \cos \theta.$$

$$\cos(90-\theta) = \frac{OM'}{OP'} = \frac{MP}{OP} = \sin \theta.$$

$$\tan(90-\theta) = \frac{M'P'}{OM'} = \frac{OM}{MP} = \cot \theta.$$

$$\cot(90-\theta) = \frac{OM'}{M'P'} = \frac{MP}{OM} = \tan\theta.$$

$$\sec(90-\theta) = \frac{OP'}{OM'} = \frac{OP}{MP} = \operatorname{cosec}\theta.$$

NOTES

$$\operatorname{cosec}(90-\theta) = \frac{OP'}{M'P'} = \frac{OP}{OM} = \sec\theta.$$

***Angle* (90 + θ):** Let a revolving line OR starting from OX in anticlockwise direction, trace out an angle $XOR = \theta$. Let another revolving line OR' starting from OX in anticlockwise direction, first trace out an angle of 90° and then revolve further through an angle θ (see Figure 1.34).

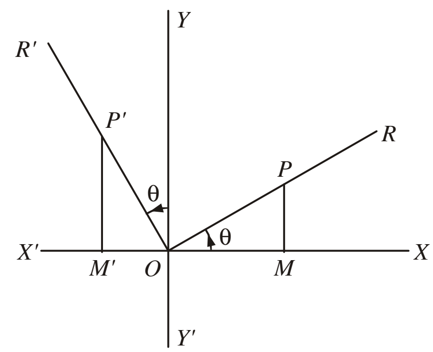


Figure 1.34 *Angle 90 + θ*

Take P and P' on OR and OR' respectively such that $OP' = OP$.

Then, OR' has traced an angle, $XOR' = 90° + \theta$.

Draw $PM \perp OX$ and $P'M' \perp OX'$.

Δs OPM and $P'OM'$ are congruent.

We have, $OM = M'P' - OM' = MP$ by convention of signs,

Then, $$\sin(90+\theta) = \frac{M'P'}{OP} = \frac{OM}{OP} = \cos\theta.$$

$$\cos(90+\theta) = \frac{OM'}{OP'} = \frac{-MP}{OP} = -\sin\theta.$$

$$\tan(90+\theta) = \frac{M'P'}{OM'} = \frac{OM}{-MP} = -\cot\theta.$$

$$\cot(90+\theta) = \frac{OM'}{M'P'} = \frac{-MP}{OM} = -\tan\theta.$$

$$\sec(90+\theta) = \frac{OP'}{OM'} = \frac{OP}{-MP} = -\operatorname{cosec}\theta.$$

$$\operatorname{cosec}(90+\theta) = \frac{OP'}{M'P'} = \frac{OP}{OM} = \sec\theta.$$

***Angle* (180 – θ):** Let a revolving line OR starting from OX in anticlockwise direction, trace out an angle $XOR = \theta$ (see Figure 1.35). Let another revolving OR' starting from OX in anticlockwise direction trace out an angle of 180° and then revolve back through an angle θ. Thus, OR' has traced an angle,

$$XOR' = 180 - \theta.$$

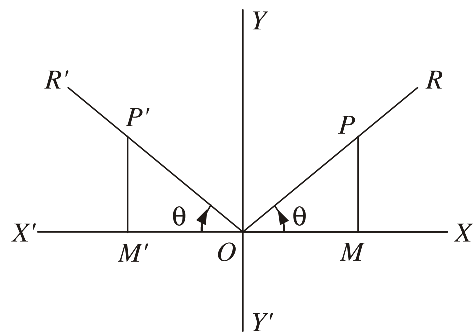


Figure 1.35 *Angle 180 – θ*

NOTES

Take P and P' on OR and OR' respectively such that $OP = OP'$.

Draw $PM \perp OX$, $P'M' \perp OX'$.

Δs OPM and $OP'M'$ are congruent.

When have $\quad OM' = -OM$ by convention of signs.

$$M'P' = MP$$

Then from $\Delta OM'P'$,

$$\sin(180 - \theta) = \frac{M'P'}{OP'} = \frac{MP}{OP} = \sin\theta.$$

$$\cos(180 - \theta) = \frac{OM'}{OP'} = \frac{-OM}{OP} = -\cos\theta.$$

$$\tan(180 - \theta) = \frac{M'P'}{OM'} = \frac{MP}{-OM} = -\tan\theta.$$

$$\cot(180 - \theta) = \frac{OM'}{M'P'} = \frac{-OM}{MP} = -\cot\theta.$$

$$\sec(180 - \theta) = \frac{OP'}{OM'} = \frac{OP}{-OM} = -\sec\theta.$$

$$\text{cosec}(180 - \theta) = \frac{OP'}{M'O'} = \frac{OP}{MP} = \text{cosec}\,\theta.$$

***Angle* (180 + θ):** Let a revolving line OR starting from OX in anticlockwise direction trace an angle $XOR = \theta$ (see Figure 1.36). Let another revolving line starting from OX in anticlockwise direction first trace out an angle of 180° and revolve further through an angle θ in the same direction. Take P and P' on OR and OR' respectively such that $OP = OP'$.

Let,

$PM \perp OX$, $P'M' \perp OX'$.

Δs OPM and $OP'M'$ are congruent.

NOTES

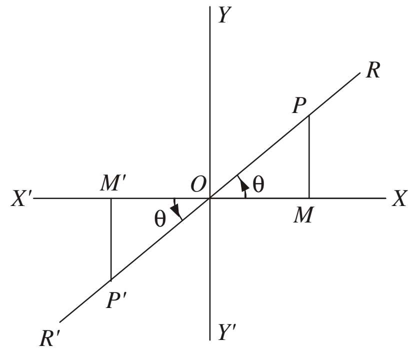


Figure 1.36 *Angle 180 + θ*

Since, $OP' = OP$

We have, $OM = -\,OM'$

$MP = -\,M'P'$ by convention of signs.

Then, $\sin(180 + \theta) = \dfrac{M'P'}{OP'} = \dfrac{-\,MP}{OP} = -\sin\theta.$

$$\cos(180 + \theta) = \frac{OM'}{OP'} = \frac{-OM}{OP} = -\cos\theta.$$

$$\tan(180 + \theta) = \frac{M'P'}{OM'} = \frac{-\,MP}{-OM} = \frac{MP}{OM} = \tan\theta.$$

$$\cot(180 + \theta) = \frac{OM'}{M'P'} = \frac{-OM}{-\,MP} = \frac{OM}{MP} = \cot\theta.$$

$$\sec(180 + \theta) = \frac{OP'}{OM'} = \frac{OP}{-OM} = -\sec\theta.$$

$$\text{cosec}(180 + \theta) = \frac{OP'}{MP'} = \frac{OP}{-\,MP} = -\,\text{cosec}\,\theta.$$

***Angle* (360 – θ):** As before let $\angle XOR = \theta$. Let another revolving line OR' starting from OX in anticlockwise direction, first trace out an angle of 360º and then, revolve back through an angle θ (see Figure 1.37). Thus, OR' has turned through an angle $XOR' = 360 - \theta$. Take P and P' on OR and OR' respectively such that $OP = OP'$.

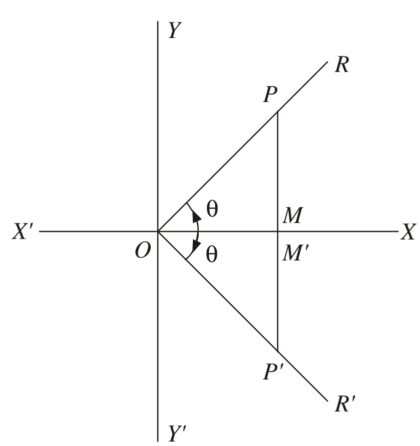


Figure 1.37 *Angle 360 – θ*

NOTES

Let $PM \perp OX$, $P'M' \perp OX$.

Δs OPM, $OP'M'$ are congruent.

Now, $OP' = OP$

We have, $OM = OM'$

$M'P' = -MP$, by convention of signs. [M and M′ coincide]

Then, $\sin(360 - \theta) = \dfrac{M'P'}{OP'} = \dfrac{-MP}{OP} = -\sin\theta.$

$$\cos(360 - \theta) = \frac{OM'}{OP'} = \frac{-OM}{OP} = \cos\theta.$$

$$\tan(360 - \theta) = \frac{M'P'}{OM'} = \frac{-MP}{OM} = -\tan\theta.$$

$$\cot(360 - \theta) = \frac{OM'}{M'P'} = \frac{OM}{-MP} = -\cot\theta.$$

$$\text{cosec}(360 - \theta) = \frac{OP'}{M'P'} = \frac{OP}{-MP} = -\text{cosec}\,\theta.$$

$$\sec(360 - \theta) = \frac{OP'}{OM'} = \frac{OP}{OM} = \sec\theta.$$

***Angle* (360° + θ):** Let a revolving line OR starting from OX in anticlockwise direction, trace out an angle $XOR = \theta$ (Figure 1.38). Let another revolving line OR' starting from OX in the same direction as OP first trace out an angle of 360° and further revolve through an angle θ. Thus OR' has traced an angle $XOR' = 360° + \theta$.

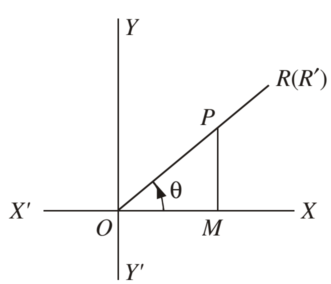


Figure 1.38 *Angle 360 + θ*

Here, OR' coincides with OR.

Thus, trigonometrical ratios of $360 + \theta$ are same as those of θ.

Note: It can be easily seen that trigonometrical ratios of $(n \times 360 \pm \theta)$ are same as those of $360 \pm \theta$, where n is any integer.

Example 1.53: Find the values of (*i*) tan (– 945°) (*ii*) sec (225°).

Solution: (*i*) $\tan(-945°) = -\tan 945°$

$$= -\tan(3 \times 360 - 135°)$$
$$= -\tan(360° - 135°)$$
$$= -[-\tan 135°] = \tan 135°$$
$$= \tan(90° + 45°) = -\cot 45° = -1.$$

(*ii*) sec 225° = sec (180° + 45°) = – sec 45° = $-\sqrt{2}$.

Example 1.54: If cos θ = a, find the values of cosec $\left(\frac{\pi}{2}+\theta\right)$ and $\sin\left(\frac{3\pi}{2}-\theta\right)$.

NOTES

Solution: Now, cosec $\left(\frac{\pi}{2}+\theta\right) = \sec\theta = \frac{1}{\cos\theta} = \frac{1}{a}$

and $$\sin\left(\frac{3\pi}{2}-\theta\right) = \sin\left(\pi+\frac{\pi}{2}-\theta\right)$$
$$= -\sin\left(\frac{\pi}{2}-\theta\right) = -\cos\theta = -a.$$

Example 1.55: Find the trigonometrical ratios of 270° – θ in terms of those of θ for all vlaues of θ.

Solution: $$\sin(270° - \theta) = \sin\left(180° + \overline{90° - \theta}\right)$$
$$= -\sin(90° - \theta) = -\cos\theta.$$
$$\cos(270° - \theta) = \cos\left(180° + \overline{90° - \theta}\right)$$
$$= -\cos(90° - \theta) = -\sin\theta.$$
$$\tan(270° - \theta) = \tan\left(180° + \overline{90° - \theta}\right)$$
$$= \tan(90° - \theta) = \cot\theta.$$
$$\cot(270° - \theta) = \cot\left(180° + \overline{90° - \theta}\right)$$
$$= \cot(90° - \theta) = \tan\theta.$$
$$\sec(270° - \theta) = \sec\left(180° + \overline{90° - \theta}\right)$$
$$= -\sec(90° - \theta) = -\operatorname{cosec}\theta.$$
$$\operatorname{cosec}(270° - \theta) = \operatorname{cosec}\left(180° + \overline{90° - \theta}\right)$$
$$= -\operatorname{cosec}(90° - \theta) = -\sec\theta.$$

Example 1.56: What value of x between 0° and 90° will satisfy the equation tan 2x tan 4x = 1?

Solution: $\tan 2x \tan 4x = 1$

$$\Rightarrow \quad \tan 2x = \frac{1}{\tan 4x} = \cot 4x$$

$$\Rightarrow \quad \tan 2x = \tan\left(\frac{\pi}{2} - 4x\right)$$

$$\Rightarrow \quad 2x = \frac{\pi}{2} - 4x, \quad \text{or} \quad 2x = \pi + \left(\frac{\pi}{2} - 4x\right),$$

or $$2x = 2\pi + \left(\frac{\pi}{2} - 4x\right)$$

$$\Rightarrow \quad x = 15°, 45°, 75°.$$

Example 1.57: Find the value of tan 5° tan 25° tan 45° tan 65° tan 85°.

Solution: Now, $\tan 85° = \tan(90° - 5°) = \cot 5°$

and $\tan 65° = \tan(90° - 25°) = \cot 25°$

So, $\tan 5° \tan 25° \tan 45° \tan 65° \tan 85°$

$$= (\tan 5° \cot 5°)(\tan 25° \cot 25°) \tan 45°$$

$$= \tan 45° = 1.$$

Example 1.58: Show that:

$$\frac{\sin(270° + \theta)\cos^3(720° - \theta) - \sin(270° - \theta)\sin^3(540° + \theta)}{\sin(90° + \theta)\sin(-\theta) - \cos^2(280° - \theta)} + \frac{\cot(270° - \theta)}{\operatorname{cosec}^2(450° + \theta)} = 1.$$

where θ is taken such as the denominator appearing in any fraction in the expression does not vanish.

Solution: Now, $\cos^3(720° - \theta) = \cos^3\theta$

$$\sin^3(540° + \theta) = -\sin^3\theta$$

$$\operatorname{cosec}^2(450° + \theta) = \sec^2\theta$$

$$\sin(270° + \theta) = -\cos\theta$$

$$\sin(270° - \theta) = -\cos\theta$$

$$\cot(270° - \theta) = \tan\theta$$

So, the given expression is equal to,

$$= \frac{-\cos\theta\cos^3\theta - \sin^3\theta\cos\theta}{-\cos\theta\sin\theta - \cos^2\theta} + \frac{\tan\theta}{\sec^2\theta}$$

$$= \frac{-\cos\theta(\sin^3\theta + \cos^3\theta)}{-\cos\theta(\sin\theta + \cos\theta)} + \frac{\sin\theta\cos\theta}{1}$$

$$= \frac{-(\sin\theta + \cos\theta)(\sin^2\theta + \cos^2\theta - \sin\theta\cos\theta)}{-(\sin\theta + \cos\theta)} + \frac{\sin\theta\cos\theta}{1}$$

$$= 1 - \sin\theta\cos\theta + \sin\theta\cos\theta = 1.$$

Example 1.59: If θ is the angle in the fourth quadrant satisfying the equation $\cot^2\theta = 4$, find the vlaue of $\frac{1}{\sqrt{5}}(\sec\theta - \operatorname{cosec}\theta)$.

Solution: $\cot^2\theta = \operatorname{cosec}^2\theta - 1$

$\Rightarrow \quad 4 = \operatorname{cosec}^2\theta - 1$

$\Rightarrow \quad \operatorname{cosec}^2\theta = 5$

$\Rightarrow \quad \operatorname{cosec}\theta = -\sqrt{5}$ (as θ lies in fourth quadrant)

Also $\cot^2\theta = 4 \quad \Rightarrow \quad \tan^2\theta = \frac{1}{4}$

So, $\tan^2\theta = \sec^2\theta - 1$

$\Rightarrow \quad \frac{1}{4} = \sec^2\theta - 1 \quad \Rightarrow \quad \sec^2\theta = \frac{5}{4}$

$\Rightarrow \quad \sec\theta = \frac{\sqrt{5}}{2}$

Therefore, $\frac{1}{\sqrt{5}}(\sec\theta - \operatorname{cosec}\theta) = \frac{1}{\sqrt{5}}\left(\frac{\sqrt{5}}{2} + \sqrt{5}\right) = \frac{3}{2}$.

NOTES

CHECK YOUR PROGRESS

10. If θ is an acute angle, find its value from $\tan\theta = 2\sin\theta$.
11. If A, B, C are angles of a triangle, prove that:
    $\cot(A + B) + \cot C = 0$.
12. In any triangle ABC, $2\sin A + \sqrt{3}\sin B = \frac{5}{2}$ and $\sqrt{3}\sin A + 2\sin B = \frac{3\sqrt{3}}{2}$. Find the angle C.
13. Given that θ is an angle between 180º and 270º. Find the value of θ if it satisfies the equation $3\cos^2\theta - \sin^2\theta = 1$.

1.6 INVERSE TRIGONOMETRIC FUNCTIONS

The inverse sine function is written as:

$$y = \sin^{-1} x, \text{ which implies that } x = \sin y,\ -\frac{1}{2}\pi \le y \le \frac{1}{2}\pi$$

Note: $\sin^{-1} x \neq (\sin x)^{-1}$

$$(\sin x)^{-1} = \frac{1}{\sin x} \text{ and } \sin(\sin^{-1} x) = x$$

$$\cos(\cos^{-1} x) = x,\ \cos^{-1}(\cos x) = x$$

$$\sin^{-1}\left(\frac{1}{\sqrt{2}}\right) = \frac{\pi}{4} \text{ since } \sin\frac{\pi}{4} = \frac{1}{\sqrt{2}}$$

$$\sin^{-1}\left(-\frac{1}{\sqrt{2}}\right) = -\frac{\pi}{4} \text{ since } \sin\left(-\frac{\pi}{4}\right) = -\sin\frac{\pi}{4} = -\frac{1}{\sqrt{2}}$$

Other trigonometric inverse functions are $\cos^{-1} x$, $\tan^{-1} x$, $\sec^{-1} x$, $\operatorname{cosec}^{-1} x$, $\cot^{-1} x$.

$$\cos^{-1}\left(\frac{1}{\sqrt{2}}\right) = \frac{\pi}{4}, \text{ since } \cos\frac{\pi}{4} = \frac{1}{\sqrt{2}}.$$

Example 1.60: Prove $\cos^{-1} x = \frac{1}{2}\pi - \sin^{-1} x$ for $|x| \le 1$.

Solution: Let $\cos^{-1} x = y$ $\quad\therefore\quad \cos y = x \quad$ or $\quad \sin\left(\frac{\pi}{2} - y\right) = x$

$\therefore\quad \sin^{-1} x = \frac{\pi}{2} - y = \frac{\pi}{2} - \cos^{-1} x.\ \Rightarrow \cos^{-1} x = \frac{\pi}{2} - \sin^{-1} x$ Hence proved.

Example 1.61: Plot the graphs of $y = \sin^{-1} x, \cos^{-1} x, \tan^{-1} x$.

Solution: Graphs of Inverse Functions are as follows (see Figure 1.39):

NOTES

If $x = 0$, $\quad \sin^{-1} x = \sin^{-1} 0 = 0, \quad \cos^{-1} 0 = \pi/2.$

If $x = \frac{1}{2}$, $\quad \sin^{-1} 1/2 = \pi/6, \quad \cos^{-1} 1/2 = \pi/3.$

If $x = -1/2$ $\quad \sin^{-1} (-1/2) = -\pi/6, \quad \cos^{-1} (-1/2) = \pi/3.$

If $x = 1$ $\quad \sin^{-1} 1 = \pi/2, \quad \cos^{-1} 1 = 0.$

If $x = -1$, $\quad \sin^{-1} (-1) = -\pi/2, \quad \cos^{-1} (-1) = \pi.$

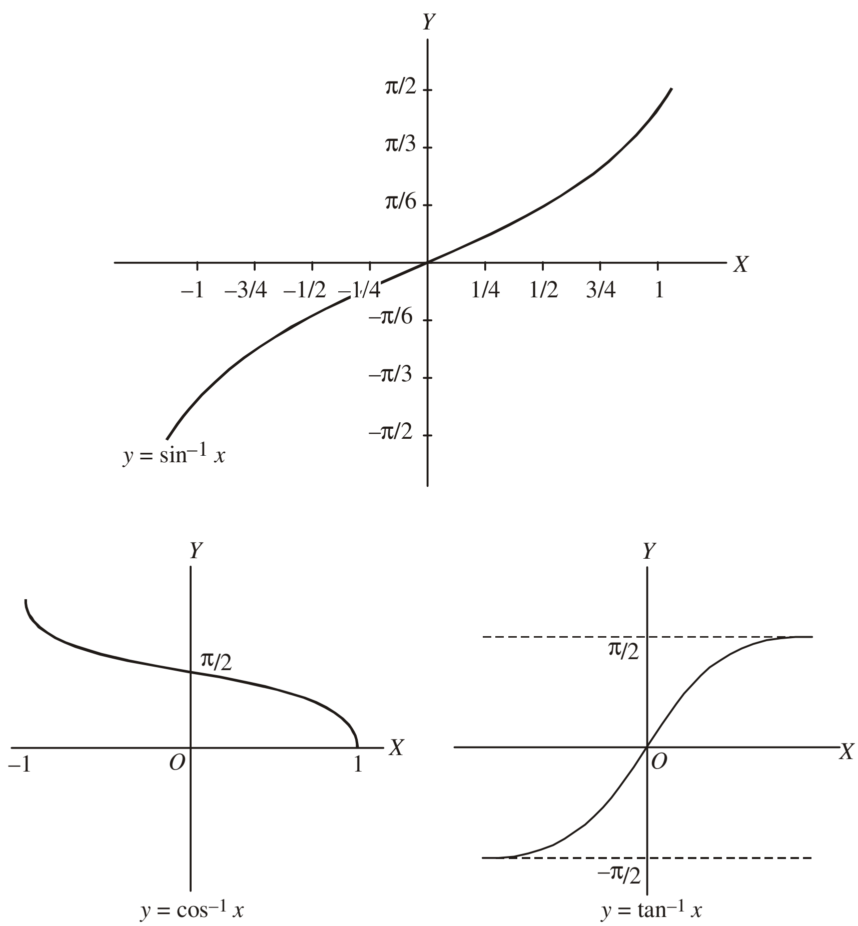


***Figure 1.39** Graphs of Inverse Functions*

1.6.1 Range of Trigonometric Functions

NOTES

$$y = \sin x$$

The range of $y = \sin x$ is $-1 \le y \le 1$. The function increases strictly from -1 to $+1$ as x increases from $-\pi/2$ to $\pi/2$ and decreases strictly from $+1$ to -1 as x increases from $\pi/2$ to $3\pi/2$ and so on (see Figure 1.40).

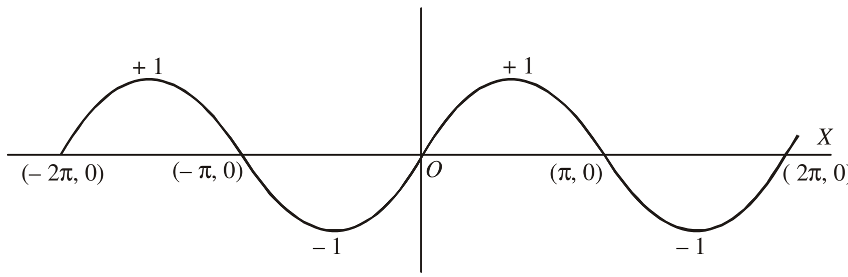


Figure 1.40 *Range of Trigonometric Functions*

The range of $y = \cos x$ is $-1 \le y \le 1$

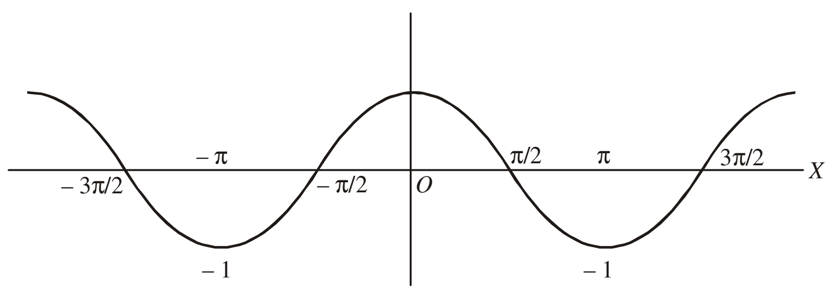


Figure 1.41 *Range of y = tan x*

The range of $y = \tan x$ is $-\infty \le y \le \infty$ (see Figure 1.41).

The range of $y = \tan^{-1} x$ is $-\dfrac{\pi}{2} \le y \le \dfrac{\pi}{2}$ (see Figure 1.42).

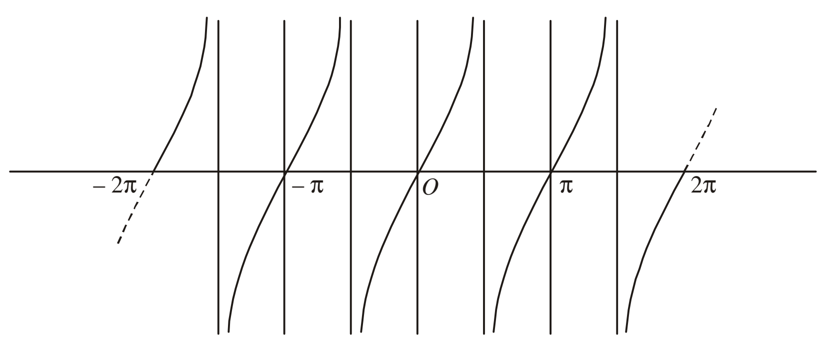


Figure 1.42 *Range of y = tan*$^{-1}$

The functions sec x, cosec x and cot x can be examined on the same lines.

Note**:** (*i*) $|\sin x| < |x|$ when x lies in $\left(-\dfrac{\pi}{2}, \dfrac{\pi}{2}\right)$. (*ii*) $\sin x < x < \tan x$.

NOTES

1.6.2 Properties of Inverse Trigonometric Functions

I. **If $\sin^{-1} x = y$, $\sin y = x$**

$\therefore \quad \sin^{-1}(\sin y) = y, \cos^{-1}(\cos y) = y$

Also, $\sin(\sin^{-1} y) = y, \cos(\cos^{-1} y) = y$

II. ***Reciprocity*** $\sin^{-1}\dfrac{1}{x} = \text{cosec}^{-1} x$

$\left(\text{Since } \sin^{-1}\dfrac{1}{x} = y \text{ implies } \sin y = \dfrac{1}{x} \text{ or } x = \dfrac{1}{\sin y}\right)$

$\therefore x = \text{cosec}\, y \quad \text{or} \quad y = \text{cosec}^{-1} x$

Similarly, $\cos^{-1}\dfrac{1}{x} = \sec^{-1} x, \qquad \sec^{-1}\dfrac{1}{x} = \cos^{-1} x$

$$\tan^{-1}\frac{1}{x} = \cot^{-1} x, \qquad \cot^{-1}\frac{1}{x} = \tan^{-1} x \qquad \text{...(1.20)}$$

III. ***Inverse functions are odd functions***

$$\sin^{-1}(-x) = -\sin^{-1} x$$

$$[\sin^{-1}(-x) = y \text{ implies } -x = \sin y \text{ or } x = -\sin y]$$

$$\tan^{-1}(-x) = -\tan^{-1} x$$

$$\text{cosec}^{-1}(-x) = -\text{cosec}^{-1} x$$

IV. ***Other properties***

$$\sin^{-1}(-x) = -\sin^{-1} x \qquad -1 \le x \le 1$$

$$\text{cosec}^{-1}(-x) = -\text{cosec}^{-1} x \qquad -1 \ge x, x \ge 1$$

$$\cos^{-1}(-x) = \pi - \cos^{-1} x \qquad -1 \le x \le 1$$

$$\sec^{-1}(-x) = \pi - \sec^{-1} x \qquad -1 \ge x, x \ge 1$$

$$\tan^{-1}(-x) = -\tan^{-1} x \qquad -\infty < x < \infty$$

$$\cot^{-1}(-x) = \pi - \cot^{-1} x \qquad -\infty < x < \infty \qquad \text{...(1.21)}$$

$$\sin^{-1} x + \cos^{-1} x = \frac{\pi}{2}$$

$$\tan^{-1} x + \cot^{-1} x = \frac{\pi}{2}$$

$$\text{cosec}^{-1} x + \sec^{-1} x = \frac{\pi}{2} \qquad \text{...(1.22)}$$

$$\tan^{-1} x + \tan^{-1} y = \tan^{-1}\frac{x+y}{1-xy}, \qquad \text{where } xy < 1, x > 0, y > 0$$

$$\tan^{-1} x - \tan^{-1} y = \tan^{-1}\frac{x-y}{1+xy} \qquad \text{...(1.23)}$$

$$x,\ y > 0$$

For example, if $\sin^{-1} x = \theta$

NOTES

$$x = \sin\theta = \cos\left(\frac{\pi}{2} - \theta\right)$$

$$\therefore \qquad \cos^{-1} x = \frac{\pi}{2} - \theta$$

$$\therefore \ \sin^{-1} x + \cos^{-1} x = \frac{\pi}{2}.$$

1.7 TRIGONOMETRIC EQUATIONS

A trigonometric equation involves trigonometric expressions like $\sin\theta$, $\cos\theta$, etc., where θ is an unknown. The solution of the equation is the value or sometimes values of θ that satisfies the equation.

Since the number of solutions of a trigonometric equation is infinite, the numerically smallest angle α is of importance, i.e., the equation $\sin\theta = k$ has solutions,

$$\theta = n\pi + (-1)^n \alpha \begin{cases} n \in I \\ n = \ldots -2, -1, 0, 1, 2, 3, \ldots \\ \text{(All integers} - \text{ve or} + \text{ve)} \end{cases}$$

Where α is numerically the smallest angle for which $\sin\alpha = k$

$$\cos\theta = k \text{ has solutions } \theta = 2n\pi \pm \alpha \ (n \in I)$$

Where α is numercially the smallest angle for which $\cos\alpha = k$.

Similarly, for $\tan\theta = k$, the solutions are $\theta = n\pi + \alpha \ (n \in I)$

Where α is numerically the smallest angle for which $\tan\alpha = k$.

Example 1.62: $2\sin^2\theta + \sqrt{3}\cos\theta + 1 = 0,\ 0 \le \theta \le \pi$

Solution:

$$2(1 - \cos^2\theta) + \sqrt{3}\cos\theta + 1 = 0 \qquad \text{i.e., } 2\cos^2\theta - \sqrt{3}\cos\theta - 3 = 0$$

$$\therefore \quad (\cos\theta - \sqrt{3})(2\cos\theta + \sqrt{3}) = 0$$

$\cos\theta = \sqrt{3}$ not possible; $\cos\theta = -\dfrac{\sqrt{3}}{2}$ or $\theta = \alpha = \dfrac{5\pi}{6}$ or $\dfrac{7\pi}{6}$.

Since θ has between 0 and π, $\dfrac{\pi}{6}$ is discarded. Hence, $\theta = \dfrac{5\pi}{6}$

Example 1.63: Given the usual trigonometric ratios, find the trigonometric ratios of 18°, 36°, 54°, 72°.

Solution: Let $A = 18°$ then, $2A = 36° = 90° - 3A$

NOTES

$$\sin 2A = \sin(90 - 3A) = \cos 3A$$

i.e., $2 \sin A \cos A = 4 \cos^3 A - 3 \cos A$

$\therefore$ $\cos A(2 \sin A - 4 \cos^2 A + 3) = 0$

Since, $\cos A \neq 0$, $2 \sin A - 4 \cos^2 A + 3 = 0$

$\therefore$ $2 \sin A - 4(1 - \sin^2 A) + 3 = 0$

$\therefore$ $4 \sin^2 A + 2 \sin A - 1 = 0$ (compare with $ax^2 + bx + c = 0$)

$\therefore$ $$\sin A = \sin 18^\circ = \frac{-2 \pm 2\sqrt{5}}{8} = \frac{-1 \pm \sqrt{5}}{4}$$

$\therefore$ $\sin 18^\circ = \dfrac{-1+\sqrt{5}}{4}$, a positive value

or $= \dfrac{-1-\sqrt{5}}{4}$, a negative value is not possible in the first quadrant.

Hence, $\sin 18^\circ = \dfrac{\sqrt{5}-1}{4}$

$\therefore$ $$\cos 18^\circ = \sqrt{1 - \sin^2 18^\circ} = \frac{\sqrt{10 + 2\sqrt{5}}}{4}$$

$$\tan 18^\circ = \frac{\sqrt{5}-1}{\sqrt{10+2\sqrt{5}}}$$

$$\sin 36^\circ = 2 \sin 18^\circ \cos 18^\circ = \frac{2}{16}(\sqrt{5}-1)(\sqrt{10+2\sqrt{5}})$$

$$= \frac{\sqrt{10-2\sqrt{5}}}{4}$$

$$\cos 36^\circ = \frac{\sqrt{5}+1}{4}$$

$$\tan 36^\circ = \frac{\sqrt{5}+1}{\sqrt{10-2\sqrt{5}}}$$

Since, $\cos(90 - 2A) = \sin 2A$

i.e., $$\cos 54^\circ = \sin 36^\circ = \frac{\sqrt{10-2\sqrt{5}}}{4}$$

$$\sin 54^\circ = \cos 36^\circ = \frac{\sqrt{5}+1}{4}$$

$$\sin 72^\circ = \cos 18^\circ = \frac{\sqrt{10+2\sqrt{5}}}{4}$$

$$\cos 72^\circ = \sin 18^\circ = \frac{\sqrt{5}-1}{4}.$$

NOTES

CHECK YOUR PROGRESS

14. Find (*i*) $\cos(\tan^{-1} 3/4)$, (*ii*) $\sin \cot^{-1} x$
15. Solve the following:
    (*i*) $2\sin^2\theta = 3\cos\theta$
    (*ii*) $4\sin^4\theta + 12\cos^2\theta = 7$
    (*iii*) $(1 - \tan\theta)(1 + \sin 2\theta) = 1 + \tan\theta$
16. Solve the following:
    (*i*) $\cos 6\theta + \cos 4\theta + \cos 2\theta + 1 = 0$
    (*ii*) $\sec 4\theta - \sec\theta = 2$
    (*iii*) $\cos(\pi/2 + 5\theta) + \sin\theta - 2\cos 3\theta + 0$
    (*iv*) $\sin 3\alpha = 4\sin\theta\sin(\theta + \alpha)\sin(\theta - \alpha)$

1.8 TRANSFORMATION OF TRIGONOMETRIC RATIOS OF SUMS, DIFFERENCES AND PRODUCTS

1. To Prove that for any Angles A and B

(1) $\sin(A + B) = \sin A\cos B + \cos A\sin B$

(2) $\cos(A + B) = \cos A\cos B - \sin A\sin B$

(3) $\tan(A + B) = \dfrac{\tan A + \tan B}{1 - \tan A\tan B}$

Proof: Let the revolving line start from OX and trace out the angle $XOY = A$ in the anticlockwise direction. Let the revolving line further trace out the angle $YOZ = B$ in the same direction (see Figure 1.43).

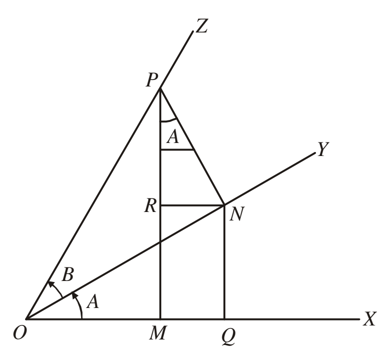


Figure 1.43 *Angles A and B in Anticlockwise Direction*

From any point P on OZ, draw $PM \perp OX$ and $PN \perp OY$. Through N draw $NR \parallel OX$ to meet MP in R.

Then, $\angle RPN = 90° - \angle PNR = \angle RNO = A$

NOTES

(1) $$\sin(A+B) = \frac{MP}{OP} = \frac{MR+RP}{OP}$$
$$= \frac{MR}{OP} + \frac{RP}{OP} = \frac{QN}{OP} + \frac{RP}{OP} \text{ (Where } NQ \perp OX\text{)}$$
$$= \frac{QN}{ON}\cdot\frac{ON}{OP} + \frac{RP}{NP}\cdot\frac{NP}{OP}$$
$$= \sin A \cos B + \cos A \sin B.$$

(2) $$\cos(A+B) = \frac{OM}{OP} = \frac{OQ-MQ}{OP}$$
$$= \frac{OQ}{OP} - \frac{MQ}{OP} = \frac{OQ}{OP} - \frac{RN}{OP}$$
$$= \frac{OQ}{ON}\cdot\frac{ON}{OP} - \frac{RN}{NP}\cdot\frac{NP}{OP}$$
$$= \cos A \cos B - \sin A \sin B.$$

(3) $$\tan(A+B) = \frac{MP}{OM} = \frac{QN+RP}{OQ-RN}$$
$$= \frac{\frac{QN}{OQ} + \frac{RP}{OQ}}{1 - \frac{RN}{OQ}} = \frac{\tan A + \frac{RP}{OQ}}{1 - \frac{RN}{RP}\cdot\frac{RP}{OQ}}.$$

Since the angles RPN and QON are equal, the triangles RPN and QON are similar, so that

$$\frac{RP}{PN} = \frac{OQ}{ON}$$

i.e. $$\frac{RP}{OQ} = \frac{PN}{ON} = \tan B.$$

Thus, $$\tan(A+B) = \frac{\tan A + \tan B}{1 - \tan A \tan B}.$$

***Note*:** The figure has been drawn for acute angles A, B and $A+B$. The result is, however, true for all angles A and B.

For, let $A_1 = 90° + A$, then $\sin A_1 = \cos A$,

$$\cos A_1 = -\sin A$$

so that $$\sin(A_1 + B) = \sin(90° + A + B)$$
$$= \cos(A+B)$$
$$= \cos A \cos B - \sin A \sin B$$
$$= \sin A_1 \cos B + \cos A_1 \sin B.$$

Similarly, if B is increased by 90°, the result is again true and so on.

2. To prove that for any angles A and B

(1) $\sin(\mathrm{A} - \mathrm{B}) = \sin \mathrm{A} \cos \mathrm{B} - \cos \mathrm{A} \sin \mathrm{B}$

(2) $\cos(\mathrm{A} - \mathrm{B}) = \cos \mathrm{A} \cos \mathrm{B} + \sin \mathrm{A} \sin \mathrm{B}$

(3) $\tan(\mathrm{A} - \mathrm{B}) = \dfrac{\tan \mathrm{A} - \tan \mathrm{B}}{1 + \tan \mathrm{A} \tan \mathrm{B}}$.

NOTES

Proof: Let the revolving line starting from OX in anticlockwise direction trace out the angle $YOX = A$ and then revolving back in clockwise direction, trace out angle $YOZ = B$. The revolving line thus traced out the angle $XOZ = A - B$ (see Figure 1.44).

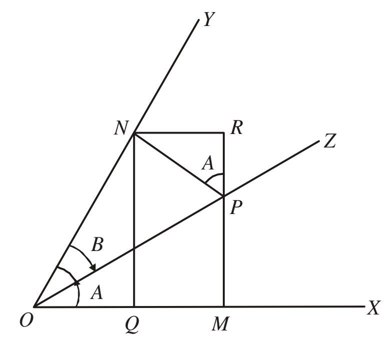


Figure 1.44 *Angles A and B in Anticlockwise and Clockwise Direction*

From any point P on OZ, draw $PM \perp OX$, $PN \perp OY$, $NQ \perp OX$. Through N draw $NR \parallel OX$ to meet MP produced at R.

Then, $\angle RPN = 90° - \angle PNR = \angle RNY = A$.

(1) $\sin (A - B) = \sin XOZ$

$$= \frac{MP}{OP} = \frac{MR - PR}{OP}$$

$$= \frac{MR}{OP} - \frac{PR}{OP} = \frac{QN}{OP} \cdot \frac{PR}{OP}$$

$$= \frac{QN}{ON}\,\frac{ON}{OP} - \frac{PR}{NP}\,\frac{NP}{OP}$$

$= \sin A \cos B - \cos A \sin B$.

(2) $\cos (A - B) = \cos XOZ$

$$= \frac{OM}{OP} = \frac{OQ + QM}{OP} = \frac{OQ}{OP} + \frac{QM}{OP}$$

$$= \frac{OQ}{OP} + \frac{NR}{OP}$$

$$= \frac{OQ}{ON} \cdot \frac{ON}{OP} + \frac{NR}{NP} \cdot \frac{NP}{OP}$$

$= \cos A \cos B + \sin A \sin B$.

(3) $\tan (A - B) = \tan XOZ$

$$= \frac{MP}{OM} = \frac{MR - PR}{OQ + QM} = \frac{QN - PR}{OQ + NR}$$

$$= \frac{\frac{QN}{OQ} - \frac{PR}{OQ}}{1 + \frac{NR}{OQ}} = \frac{\tan A - \frac{PR}{OQ}}{1 + \frac{NR}{PR} \cdot \frac{PR}{OQ}}$$

Since the angles RPN and NOQ are equal, the triangles RPN and NOQ are similar, so that

$$\frac{PR}{OQ} = \frac{PN}{ON} = \tan B$$

Thus tan, $(A - B) = \dfrac{\tan A - \tan B}{1 + \tan A \tan B}$

Note: The figure in this article is again drawn for acute angles A, B and $A + B$, but the result can be proved for all angles A and B.

Trigonometrical Ratios of Multiple and Submultiple Angles to prove that:

(1) (*i*) $\sin 2\text{A} = 2 \sin \text{A} \cos \text{A}$

(*ii*) $\cos 2\text{A} = 2\cos^2 \text{A} - 1 = 1 - 2\sin^2 \text{A} = \cos^2 \text{A} - \sin^2 \text{A}$

(*iii*) $\tan 2\text{A} = \dfrac{2\tan \text{A}}{1 - \tan^2 \text{A}}$

Proof: $\sin(A + B) = \sin A \cos B + \cos A \sin B$.

Put $A = B$

We have, $\sin 2A = \sin A \cos A + \cos A \sin A$

$= 2 \sin A \cos A$.

(*ii*) Again, $\cos(A + B) = \cos A \cos B - \sin A \sin B$

Put $A = B$

We have, $\cos 2A = \cos A \cos A - \sin A \sin A$

$= \cos^2 A - \sin^2 A$.

Also, $\cos^2 A - \sin^2 A = (1 - \sin^2 A) - \sin^2 A$

$= 1 - 2\sin^2 A$.

and $\cos^2 A - \sin^2 A - \cos^2 A - (1 - \cos^2 A) = 2\cos^2 A - 1$.

(*iii*) Also, $\tan(A + B) = \dfrac{\tan A + \tan B}{1 - \tan A \tan B}$.

Put $A = B$

We have, $\tan 2A = \dfrac{\tan A + \tan A}{1 - \tan A \tan A} = \dfrac{2\tan A}{1 - \tan^2 A}$.

(2)(*i*) $\sin 3\text{A} = 3\sin \text{A} - 4\sin^3 \text{A}$

(*ii*) $\cos 3\text{A} = 4\cos^3 \text{A} - 3\cos \text{A}$

(*iii*) $\tan 3\text{A} = \dfrac{3\tan \text{A} - \tan^3 \text{A}}{1 - 3\tan^2 \text{A}}$

Proof: (*i*) $\sin 3A = \sin(A + 2A)$

$= \sin A \cos 2A + \cos A \sin 2A$

$= \sin A (1 - 2\sin^2 A) + \cos A (2 \sin A \cos A)$

$= \sin A - 2\sin^3 A + 2 \sin A (1 - \sin^2 A)$

$= 3 \sin A - 4\sin^3 A$.

NOTES

(*ii*) $\cos 3A = \cos (A + 2A)$

$= \cos A \cos 2A - \sin A \sin 2A$

$= \cos A\,(2\cos^2 A - 1) - \sin A\,(2 \sin A \cos A)$

$= 2\cos^3 A - \cos A - 2\cos A\,(1 - \cos^2 A)$

$= 4\cos^3 A - 3\cos A.$

(*iii*) $\tan 3A = \tan (A + 2A)$

$$= \frac{\tan A + \tan 2A}{1 - \tan A \tan 2A} = \frac{\tan A + \dfrac{2\tan A}{1-\tan^2 A}}{1 - \tan A\,\dfrac{2\tan A}{1-\tan^2 A}}$$

$$= \frac{\tan A\,(1-\tan^2 A) + 2\tan A}{(1-\tan^2 A) - 2\tan^2 A}$$

$$= \frac{3\tan A - \tan^3 A}{1 - 3\tan^2 A}.$$

(3) (*i*) $\sin A = 2\sin\frac{A}{2}\cos\frac{A}{2}$

(*ii*) $\cos A = \cos^2\frac{A}{2} - \sin^2\frac{A}{2}$

$= 2\cos^2\frac{A}{2} - 1 = 1 - 2\sin^2\frac{A}{2}$

(*iii*) $\tan A = \dfrac{2\tan\frac{A}{2}}{1 - \tan^2\frac{A}{2}}$

Proof: Replace A by $\frac{A}{2}$ in (1) of this subsection. All relations in (3) will immediately follow.

(4) (*i*) $\sin\frac{A}{2} = \pm\sqrt{\dfrac{1-\cos A}{2}}$

(*ii*) $\cos\frac{A}{2} = \pm\sqrt{\dfrac{1+\cos A}{2}}$

(*iii*) $\tan\frac{A}{2} = \pm\sqrt{\dfrac{1-\cos A}{1+\cos A}}$

Proof: We have $\cos A = 1 - 2\sin^2\frac{A}{2}$, form (3)

Then, $2\sin^2\frac{A}{2} = 1 - \cos A$

so that $\sin^2\frac{A}{2} = \dfrac{1-\cos A}{2} \Rightarrow \sin\frac{A}{2} = \pm\sqrt{\dfrac{1-\cos A}{2}}$

NOTES

Similarly, $\cos A = 2\cos^2 \frac{A}{2} - 1$

$\Rightarrow \quad 2\cos^2 \frac{A}{2} = 1 + \cos A$

$\Rightarrow \quad 2\cos^2 \frac{A}{2} = \frac{1+\cos A}{2}$

$\Rightarrow \quad \cos \frac{A}{2} = \pm\sqrt{\frac{1+\cos A}{2}}$

Hence, $\tan \frac{A}{2} = \frac{\sin \frac{A}{2}}{\cos \frac{A}{2}} = \pm\sqrt{\frac{1-\cos A}{1+\cos A}}$

Example 1.64: If $\sin \alpha = \frac{3}{5}$ and $\cos \beta = \frac{9}{41}$, find the value of $\sin(\alpha - \beta)$, α, β being acute angles.

Solution: $\cos^2 \alpha = 1 - \sin^2 \alpha = 1 - \frac{9}{25} = \frac{16}{25}$

$\Rightarrow \quad \cos^2 \alpha = \pm\frac{4}{5}$

Since α is acute angle, $\cos \alpha = \frac{4}{5}$

Also, $\sin^2 \beta = 1 - \cos^2 \beta$

$= 1 - \frac{81}{1681} = \frac{1600}{1681}$

$\Rightarrow \quad \sin \beta = \pm\frac{40}{41}$

β is acute angle $\Rightarrow \sin \beta = \frac{40}{41}$

Therefore, $\sin(\alpha - \beta) = \sin \alpha \cos \beta - \sin \beta \cos \alpha$

$= \frac{3}{5} \times \frac{9}{41} - \frac{40}{41} \times \frac{4}{5} = \frac{27-160}{205} = \frac{133}{205}$

Example 1.65: Find the value of sin 18° and cos 18°.

Solution: Put $18° = x$ so that $90° = 5x$

Then $2x = 90° - 3x$

$\Rightarrow \quad \sin 2x = \sin(90° - 3x) = \cos 3x$

or $\quad 2\sin x \cos x = 4\cos^3 x - 3\cos x$

or $\quad 4\cos^3 x - 2\sin x \cos x - 3\cos x = 0$

or $\quad \cos x(4\cos^2 x - 2\sin x - 3) = 0$

or $\quad 4\cos^2 x - 2\sin x - 3 = 0 \quad$ since $\cos x \neq 0$

or $\quad 4(1 - \sin^2 x) - 2\sin x - 3 = 0$

NOTES

or $\quad 4\sin^2 x + 2\sin x - 1 = 0$

$$\Rightarrow \quad \sin x = \frac{-2 \pm \sqrt{20}}{8} = \frac{-1+\sqrt{5}}{4} \text{ or } \frac{-1-\sqrt{5}}{4}.$$

Since, $x = 18°$. $\sin x$ is positive so that the value $\frac{-1-\sqrt{5}}{4}$ is rejected.

Hence, $\quad \sin x = \frac{-1+\sqrt{5}}{4} \quad$ *i.e.* $\quad \sin 18° = \frac{-1+\sqrt{5}}{4}$

Also, $\quad \cos 18° = \sqrt{1-\sin^2 18°} = \sqrt{1-\left(\frac{5+1-2\sqrt{5}}{16}\right)}$

$$= \sqrt{1-\frac{16-6+2\sqrt{5}}{16}} = \sqrt{\frac{10+2\sqrt{5}}{4}}.$$

Example 1.66: If $\tan\theta = \frac{12}{5}$ and θ is in third quadrant, find the value of $2\sin\frac{\theta}{2} - 3\cos\frac{\theta}{2}$.

Solution: θ lies in third quadrant

$\Rightarrow \quad \theta$ lies between 180° and 270°

$\Rightarrow \quad \frac{\theta}{2}$ lies between 90° and 135°

$\Rightarrow \quad \sin\frac{\theta}{2}$ is positive and $\cos\frac{\theta}{2}$ is negative.

Now, $\quad \tan\theta = \frac{12}{5} \Rightarrow \sin\theta = \frac{-12}{13}$ and $\cos\theta = \frac{-5}{13}$

(as θ lies in 3rd quadrant)

Therefore, $\quad \cos\frac{\theta}{2} = -\sqrt{\frac{1+\cos\theta}{2}} = -\sqrt{\frac{8}{26}}$

and $\sin\frac{\theta}{2} = \sqrt{\frac{1-\cos\theta}{2}} = \sqrt{\frac{18}{26}}$

So $2\sin\frac{\theta}{2} - 3\cos\frac{\theta}{2} = 2\sqrt{\frac{18}{26}} + 3\sqrt{\frac{8}{26}} = 2\sqrt{\frac{9}{13}} + 3\sqrt{\frac{4}{13}}$

$$= \frac{1}{\sqrt{13}}[6+6] = \frac{12}{\sqrt{13}}.$$

Example 1.67: Prove that $\sin 5\theta = 5\sin\theta - 20\sin^3\theta + 16\sin^5\theta$.

Solution: LHS $= \sin 5\theta$

$= \sin(2\theta + 3\theta)$

$= \sin 2\theta\cos 3\theta + \cos 2\theta \sin 3\theta$

$= 2\sin\theta\cos\theta(4\cos^3\theta - 3\cos\theta)$

$\quad + (3\sin\theta - 4\sin^3\theta)(1-2\sin^2\theta)$

$= 2\sin\theta\cos^2\theta(4\cos^2\theta - 3)$

$\quad + \sin\theta(3-4\sin^2\theta)(1-2\sin^2\theta)$

$= \sin\theta\,[2(1-\sin^2\theta)(1-4\sin^2\theta)$
$\qquad\qquad + (3-4\sin^2\theta)(1-2\sin^2\theta)$
$= \sin\theta\,[2-10\sin^2\theta+8\sin^4\theta+3-10\sin^2\theta+8\sin^2\theta]$
$= 5\sin\theta-20\sin^3\theta+16\sin^5\theta.$

NOTES

Example 1.68: Prove that $\dfrac{\sin A+\sin 3A}{\cos A+\cos 3A} = \tan 2A.$

Solution: LHS= $\dfrac{\sin A+\sin 3A}{\cos A+\cos 3A} = \dfrac{\sin A+3\sin A-4\sin^3 A}{\cos A+4\cos^3 A-3\cos A}$

$$= \frac{4\sin A\,(1-\sin^2 A)}{2\cos A\,(2\cos^2 A-1)}$$

$$= \frac{2\sin A\cos^2 A}{\cos A\cos 2A} = \frac{2\sin A\cos A}{\cos 2A}$$

$$= \frac{\sin 2A}{\cos 2A} = \tan 2A.$$

Example 1.69: Prove that $1+\cos^2 2\theta = 2(\cos^4\theta+\sin^4\theta)$.

Solution: RHS $= 2(\cos^4\theta+\sin^4\theta)$

$= 2[(\cos^2\theta-\sin^2\theta)^2+2\sin^2\theta\cos^2\theta]$
$= 2[(\cos 2\theta)^2+2\sin^2\theta\cos^2\theta]$
$= 2(\cos 2\theta)^2+4\sin^2\theta\cos^2\theta$
$= 2(\cos 2\theta)^2+(\sin 2\theta)^2$
$= (\cos 2\theta)^2+[(\cos 2\theta)^2+(\sin 2\theta)^2]$
$= (\cos 2\theta)^2+1 = \text{LHS}.$

Example 1.70: If $\sin\theta+\sin\phi = a$, $\cos\theta+\cos\phi = b$, find the value of $\tan\dfrac{\theta-\phi}{2}$.

Solution: $\sin\theta+\sin\phi = a$

$\cos\theta+\cos\phi = b$

Squaring and adding, we get

$(\sin^2\theta+\cos^2\theta)+(\sin^2\phi+\cos^2\phi)+2(\sin\theta\sin\phi+\cos\theta\cos\phi) = a^2+b^2$

$\Rightarrow 2\{1+\cos(\theta-\phi)\} = a^2+b^2$

$$\Rightarrow \quad 1+\cos(\theta-\phi) = \frac{a^2+b^2}{2}$$

Also, $1-\cos(\theta-\phi) = 1-\left(\dfrac{a^2+b^2}{2}-1\right)$

$$= 2-\frac{a^2+b^2}{2} = \frac{4-a^2-b^2}{2}$$

$$\Rightarrow \qquad \tan\frac{\theta-\phi}{2} = \pm\sqrt{\frac{1-\cos(\theta-\phi)}{1+\cos(\theta-\phi)}} = \pm\sqrt{\frac{4-a^2-b^2}{a^2+b^2}}$$

NOTES

Example 1.71: Find the value of sin 9° and cos 9°.

Solution: Now, $\cos 2\theta = 1 - 2\sin^2\theta$

Put $\theta = 9°$, we get

$$\cos 18° = 1 - 2\sin^2 9°$$

$$\cos 18° = \frac{\sqrt{10+2\sqrt{5}}}{4}$$

So $$\frac{\sqrt{10+2\sqrt{5}}}{4} = 1 - 2\sin^2 9°$$

or $$2\sin^2 9° = 1 - \frac{\sqrt{10+2\sqrt{5}}}{4} = \frac{4-\sqrt{10+2\sqrt{5}}}{4}$$

or $$\sin^2 9° = \frac{4-\sqrt{10+2\sqrt{5}}}{8}$$

or $$\sin 9° = \pm\sqrt{\frac{4-\sqrt{10+2\sqrt{5}}}{8}}$$

Sin 9° is positive, so that the negative value is rejected.

Hence, $$\sin 9° = +\sqrt{\frac{4-\sqrt{10+2\sqrt{5}}}{8}}$$

Also, $$\cos 9° = \sqrt{1-\sin^2 9°} = \sqrt{1-\left(\frac{4-\sqrt{10+2\sqrt{5}}}{8}\right)}$$

$$= \sqrt{\frac{8-4+\sqrt{10+2\sqrt{5}}}{8}} = \sqrt{\frac{4+\sqrt{10+2\sqrt{5}}}{8}}$$

Example 1.72: Prove that $\dfrac{\tan A + \sec A - 1}{\tan A - \sec A + 1} = \dfrac{1+\sin A}{\cos A}$.

Solution: LHS = $$\frac{\tan A + \sec A - 1}{\tan A - \sec A + 1} = \frac{\dfrac{\sin A}{\cos A} + \dfrac{1}{\cos A} - 1}{\dfrac{\sin A}{\cos A} - \dfrac{1}{\cos A} + 1}$$

$$= \frac{\sin A + 1 - \cos A}{\sin A - 1 + \cos A} = \frac{\sin A + 2\sin^2\dfrac{A}{2}}{\sin A - 2\sin^2\dfrac{A}{2}}$$

$$= \frac{2\sin\dfrac{A}{2}\cos\dfrac{A}{2} + 2\sin^2\dfrac{A}{2}}{2\sin\dfrac{A}{2}\cos\dfrac{A}{2} - 2\sin^2\dfrac{A}{2}}$$

$$= \frac{\cos\frac{A}{2} + \sin\frac{A}{2}}{\cos\frac{A}{2} - \sin\frac{A}{2}}$$

$$= \frac{\left(\cos\frac{A}{2} + \sin\frac{A}{2}\right)\left(\cos\frac{A}{2} + \sin\frac{A}{2}\right)}{\left(\cos\frac{A}{2} - \sin\frac{A}{2}\right)\left(\cos\frac{A}{2} + \sin\frac{A}{2}\right)}$$

$$= \frac{\cos^2\frac{A}{2} + \sin^2\frac{A}{2} + 2\sin\frac{A}{2}\cos\frac{A}{2}}{\cos^2\frac{A}{2} - \sin^2\frac{A}{2}}$$

$$= \frac{1+\sin A}{\cos A} = \text{RHS}.$$

NOTES

Example 1.73: Prove that:

$$\tan\left(45° + \frac{\theta}{2}\right) = \sqrt{\frac{1+\sin\theta}{1-\sin\theta}}, \text{ where } 0 < \theta < 90°.$$

Solution: LHS = $\tan\left(45° + \frac{\theta}{2}\right)$

$$= \frac{\tan 45° + \tan\frac{\theta}{2}}{1 - \tan 45° \tan\frac{\theta}{2}} = \frac{1+\tan\frac{\theta}{2}}{1-\tan\frac{\theta}{2}}$$

$$= \frac{\cos\frac{\theta}{2} + \sin\frac{\theta}{2}}{\cos\frac{\theta}{2} - \sin\frac{\theta}{2}} = \frac{\left(\cos\frac{\theta}{2} + \sin\frac{\theta}{2}\right)^2}{\cos^2\frac{\theta}{2} - \sin^2\frac{\theta}{2}}$$

$$= \frac{1 + 2\sin\frac{\theta}{2}\cos\frac{\theta}{2}}{\cos\theta} = \frac{1+\sin\theta}{\pm\sqrt{1-\sin^2\theta}}$$

$$= \frac{1+\sin\theta}{\pm\sqrt{(1-\sin\theta)(1+\sin\theta)}} = \pm\sqrt{\frac{1+\sin\theta}{1-\sin\theta}}$$

Since $0 < \theta < 90°$, we have $\frac{\theta}{2} < 45°$, so that $\tan\left(45° + \frac{\theta}{2}\right)$ is positive.

Thus, $\tan\left(45° + \frac{\theta}{2}\right) = \sqrt{\frac{1+\sin\theta}{1-\sin\theta}}$ = RHS.

Example 1.74: Prove that

$$\frac{\cos 15° + \sin 15°}{\cos 15° - \sin 15°} = \sqrt{3}$$

NOTES

Solution: LHS $= \dfrac{1+\tan 15°}{1-\tan 15°} = \dfrac{\tan 45 + \tan 15}{1 - \tan 45 \tan 15}$

$$= \tan(45+15) = \tan 60° = \sqrt{3} = \text{RHS}.$$

Example. 75: Prove that:

$$\tan 2A - \tan A = \frac{2\sin A}{\cos A + \cos 3A}.$$

Solution: LHS $= \dfrac{\sin 2A}{\cos 2A} - \dfrac{\sin A}{\cos A}$

$$= \frac{\sin 2A \cos A - \sin A \cos 2A}{\cos 2A \cos A}$$

$$= \frac{\sin(2A - A)}{\cos 2A \cos A} = \frac{\sin A}{\cos 2A \cos A}$$

$$= \frac{2\sin A}{2\cos A \cos 2A}$$

$$= \frac{2\sin A}{2\cos A(2\cos^2 A - 1)}$$

$$= \frac{2\sin A}{4\cos^3 A - 2\cos A}$$

$$= \frac{2\sin A}{4\cos^3 A - 3\cos A + \cos A}$$

$$= \frac{2\sin A}{\cos 3A + \cos A}$$

$$= \frac{2\sin A}{\cos A + \cos 3A}$$

$$= \text{RHS}$$

Hence, the result follows.

Example 1.76: Prove that:

$$\cot\alpha = \tan\alpha + 2\tan 2\alpha + 4\tan 4\alpha + 8\cot 8\alpha.$$

Solution: Consider $\cot\alpha - \tan\alpha - 2\tan 2\alpha - 4\tan 4\alpha - 8\cot 8\alpha$

$$= \left(\frac{\cos\alpha}{\sin\alpha} - \frac{\sin\alpha}{\cos\alpha}\right) - 2\tan 2\alpha - 4\tan 4\alpha - 8\cot 8\alpha$$

$$= \frac{\cos 2\alpha}{\sin\alpha\cos\alpha} - 2\tan 2\alpha - 4\tan 4\alpha - 8\cot 8\alpha$$

$$= \frac{2\cos 2\alpha}{\sin 2\alpha} - 2\frac{\sin 2\alpha}{\cos 2\alpha} - 4\tan 4\alpha - 8\cot 8\alpha$$

$$= 2\left(\frac{\cos^2 2\alpha - \sin^2 2\alpha}{\sin 2\alpha\cos 2\alpha}\right) - 4\tan 4\alpha - 8\cot 8\alpha$$

$$= 2\times 2\frac{\cos 4\alpha}{\sin 4\alpha} - 4\frac{\sin 4\alpha}{\cos 4\alpha} - 8\cot 8\alpha$$

$$= 4\left(\frac{\cos^2 4\alpha - \sin^2 4\alpha}{\sin 4\alpha\cos 4\alpha}\right) - 8\cot 8\alpha$$

$$= 4 \times 2 \frac{\cos 8\alpha}{\sin 8\alpha} - 8 \cot 8\alpha = 8 \cot 8\alpha - 8 \cot 8\alpha = 0$$

This proves the result.

NOTES

Example 1.77: Given that:

$$\tan A + \tan B = p \textit{ and } \tan A \tan B = q$$

find $\sin^2 (A + B)$ and $\cos 2 (A + B)$.

Solution: $\sin^2 (A + B) = \dfrac{1}{\operatorname{cosec}^2 (A + B)} = \dfrac{1}{1 + \cot^2 (A + B)}$

$$= \frac{1}{1 + \frac{1}{\tan^2 (A + B)}} = \frac{\tan^2 (A + B)}{1 + \tan^2 (A + B)}$$

$$= \frac{\frac{p^2}{(1 - q)^2}}{1 + \frac{p^2}{(1 - q)^2}} = \frac{p^2}{p^2 + (1 - q)^2}$$

and $\cos 2 (A + B) = \cos^2 (A + B) - \sin^2 (A + B)$

$$= \frac{\cos^2 (A + B) - \sin^2 (A + B)}{\cos^2 (A + B) + \sin^2 (A + B)}$$

$$= \frac{1 - \tan^2 (A + B)}{1 + \tan^2 (A + B)} = \frac{1 - \frac{p^2}{(1 - q)^2}}{1 + \frac{p^2}{(1 - q)^2}}$$

$$= \frac{(1 - q)^2 - p^2}{(1 - q)^2 + p^2}.$$

Example 1.78: If $\sec^2 A - 2 \tan^2 B = 2$, prove that:

$$2 \cos 2A - \cos 2B + 1 = 0.$$

Solution: $\sec^2 A - 2 \tan^2 B = 2$

$\Rightarrow \quad \dfrac{1}{\cos^2 A} - \dfrac{2 \sin^2 B}{\cos^2 B} = 2$

$\Rightarrow \quad \cos^2 B - 2 \sin^2 B \cos^2 A = 2 \cos^2 A \cos^2 B$

$\Rightarrow \quad \cos^2 B = 2 \cos^2 A (\cos^2 B + \sin^2 B) = 2 \cos^2 A$

Now, $2 \cos 2A - \cos 2B + 1$

$= 2 (2 \cos^2 A - 1) - (2 \cos^2 B - 1) + 1$

$= 2 (\cos^2 B - 1) - (2 \cos^2 B - 1) + 1$

$= 2 \cos^2 B - 2 - 2 \cos^2 B + 1 + 1$

$= 0.$

This proves the result.

NOTES

Example 1.79: Given that:

$$\cos 45° = \frac{1}{\sqrt{2}}. \text{ Show that:}$$

$$\sin\left(292\frac{1}{2}\right)^\circ = -\frac{1}{2}\sqrt{2+\sqrt{2}}.$$

Solution: $\sin\left(292\frac{1}{2}\right)^\circ = \sin\left(270° + 22\frac{1}{2}^\circ\right) = -\cos 22\frac{1}{2}^\circ$

Now $\qquad \cos 45° = 2\cos^2\left(22\frac{1}{2}\right)^\circ - 1$

$\Rightarrow \qquad \frac{1}{\sqrt{2}} = 2\cos^2\left(22\frac{1}{2}\right)^\circ - 1$

$\Rightarrow \quad 2\cos^2\left(22\frac{1}{2}\right)^\circ = 1 + \frac{1}{\sqrt{2}} = \frac{\sqrt{2}+1}{\sqrt{2}}$

$\Rightarrow \quad \cos\left(22\frac{1}{2}\right)^\circ = \sqrt{\frac{\sqrt{2}+1}{2\sqrt{2}}} = \frac{1}{2}\sqrt{2+\sqrt{2}}$

$\Rightarrow \quad \sin\left(292\frac{1}{2}\right)^\circ = -\frac{1}{2}\sqrt{2+\sqrt{2}}.$

This proves the result.

Example 1.80: ABC is an acute-angled triangle inscribed in a circle of centre *O* and radius *OA* (= *10 cm*). If $\cos \angle BOC = \frac{4}{5}$, calculate:

(*i*) $\sin \angle BAC$

(*ii*) The length of *BC*

(*iii*) $\cos \angle OBC$

Solution:

(*i*) As per Figure 1.45, $\sin \angle BAC = \angle BOC$

$\Rightarrow \quad \cos 2\angle BAC = \cos \angle BOC = \frac{4}{5}$

$\Rightarrow 2\cos^2 \angle BAC - 1 = \frac{4}{5}$

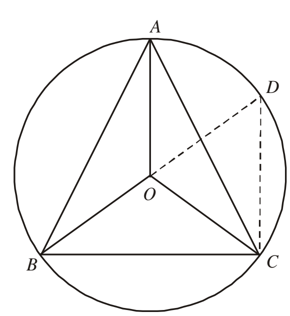


Figure 1.45

NOTES

$$\Rightarrow 2\cos^2 \angle BAC = \frac{9}{5}$$

$$\Rightarrow \cos^2 \angle BAC = \frac{9}{10}$$

$$\Rightarrow \sin^2\angle BAC = 1 - \frac{9}{10} = \frac{1}{10}$$

$$\Rightarrow \sin \angle BAC = \frac{1}{\sqrt{10}} \qquad \text{(as } BAC \text{ is an acute angle).}$$

(ii) Produce BO to meet the circle at D and join CD.

Then, BCD is a right angle (angle in a semi-circle). Also, $\angle BDC = \angle BAC$ (angles in same segment).

So, $$\sin \angle BDC = \frac{BC}{BD} = \frac{BC}{20}$$

$$\Rightarrow \sin \angle BAC = \frac{BC}{20}$$

$$\Rightarrow \frac{1}{\sqrt{10}} = \frac{BC}{20} \Rightarrow BC = \frac{20}{\sqrt{10}} = 2\sqrt{10}\,.$$

(iii) $\cos \angle OBC = \cos (90° - \angle BDC) = \cos (90° - \angle BAC)$

$$= \sin \angle BAC = \frac{1}{\sqrt{10}}.$$

Example 1.81: If tan A tan $2A \neq -1$ prove that:

$\tan 3A \tan 2A \tan A = \tan 3A - \tan 2A - \tan A$.

Solution: Now, $\tan 3A \tan 2A \tan A - \tan 3A$

$$= \tan 3A (\tan 2A \tan A - 1)$$

$$= \tan (2A + A) (\tan 2A \tan A - 1)$$

$$= \frac{\tan 2A + \tan A}{1 - \tan A \tan 2A} (\tan A \tan 2A - 1)$$

$$= -(\tan 2A + \tan A)$$

This proves the result.

Example 1.82: Prove that:

$$\frac{1 + \sin A - \cos A}{1 + \sin A + \cos A} = \tan \frac{A}{2}$$

Solution: $$\text{LHS} = \frac{(1 - \cos A) + \sin A}{(1 + \cos A) + \sin A}$$

$$= \frac{2\sin^2 \frac{A}{2} + 2\sin \frac{A}{2} \cos \frac{A}{2}}{2\cos^2 \frac{A}{2} + 2\sin \frac{A}{2} \cos \frac{A}{2}}$$

$$= \frac{2\sin \frac{A}{2}\left(\sin \frac{A}{2} + \cos \frac{A}{2}\right)}{2\cos \frac{A}{2}\left(\sin \frac{A}{2} + \cos \frac{A}{2}\right)} = \tan \frac{A}{2}$$

Example 1.83: If $\tan x = \dfrac{1-\cos y}{\sin y}$, then prove that one of the solutions will be $y = 2x$. Use this result to prove that $\tan 7\dfrac{1}{2}^{\circ} = \sqrt{6}-\sqrt{3}+\sqrt{2}-2$.

NOTES

Solution: Now, $\dfrac{1-\cos y}{\sin y} = \dfrac{1-\cos 2x}{\sin 2x} = \dfrac{2\sin^2 x}{2\sin x\cos x} = \tan x$

$\Rightarrow y = 2x$ is one of the solution.

Thus, $\tan 7\dfrac{1}{2}^{\circ} = \dfrac{1-\cos 15^{\circ}}{\sin 15^{\circ}}$

But $\cos 15^{\circ} = \cos(45-30) = \dfrac{\sqrt{3}}{2}\dfrac{1}{\sqrt{2}}+\dfrac{1}{\sqrt{2}}\dfrac{1}{2} = \dfrac{\sqrt{3}+1}{2\sqrt{2}}$

and $\sin 15^{\circ} = \sin(45-30) = \dfrac{1}{\sqrt{2}}\dfrac{\sqrt{3}}{2}-\dfrac{1}{2}\dfrac{1}{\sqrt{2}} = \dfrac{\sqrt{3}-1}{2\sqrt{2}}$

$$\Rightarrow \quad \tan 7\frac{1}{2}^{\circ} = \frac{1-\dfrac{\sqrt{3}+1}{2\sqrt{2}}}{\dfrac{\sqrt{3}-1}{2\sqrt{2}}} = \frac{2\sqrt{2}-\sqrt{3}-1}{\sqrt{3}-1}$$

$$= \frac{(2\sqrt{2}-\sqrt{3}-1)(\sqrt{3}+1)}{2}$$

$$= \frac{2\sqrt{6}-2\sqrt{3}+2\sqrt{2}-4}{2} = \sqrt{6}-\sqrt{3}+\sqrt{2}-2$$

Example 1.84: Prove that:

$$\left(1+\cos\frac{\pi}{8}\right)\left(1+\cos\frac{3\pi}{8}\right)\left(1+\cos\frac{5\pi}{8}\right)\left(1+\cos\frac{7\pi}{8}\right) = \frac{1}{8}$$

Solution: Now, $\cos\dfrac{3\pi}{8} = \cos\left(\dfrac{\pi}{2}-\dfrac{\pi}{8}\right) = -\sin\dfrac{\pi}{8}$

$$\cos\frac{5\pi}{8} = \cos\left(\frac{\pi}{2}+\frac{\pi}{8}\right) = -\sin\frac{\pi}{8}$$

$$\cos\frac{7\pi}{8} = \cos\left(\pi-\frac{\pi}{8}\right) = -\cos\frac{\pi}{8}$$

$$\text{LHS} = \left(1+\cos\frac{\pi}{8}\right)\left(1+\sin\frac{\pi}{8}\right)\left(1-\sin\frac{\pi}{8}\right)\left(1-\cos\frac{\pi}{8}\right)$$

$$= \left(1-\cos^2\frac{\pi}{8}\right)\left(1-\sin^2\frac{\pi}{8}\right)$$

$$= \sin^2\frac{\pi}{8}\cos^2\frac{\pi}{8}$$

$$= \frac{1}{4}\left(2\sin\frac{\pi}{8}\cos\frac{\pi}{8}\right)^2 = \frac{1}{4}\left(\sin\frac{\pi}{4}\right)^2$$

$$= \frac{1}{4}\left[\frac{1}{\sqrt{2}}\right]^2 = \frac{1}{8} = \text{RHS.}$$

NOTES

Example 1.85: Prove that:

$$\text{cosec}^6\,\alpha - \cot^6\alpha = 3\,\text{cosec}^2\,\alpha\cot^2\alpha + 1$$

Solution: LHS $= \text{cosec}^6\,\alpha - \cot^6\alpha = (\text{cosec}^2\,\alpha) - (\cot^2\propto)^3$

$$= (\text{cosec}^2\,\alpha - \cot^2\alpha)\,(\text{cosec}^4\,\alpha + \text{cosec}^2\,\alpha\cot^2\alpha + \cot^4\alpha)$$
$$= (\text{cosec}^4\,\alpha + \text{cosec}^2\,\alpha\cot^2\alpha + \cot^4\alpha)$$
$$= (\text{cosec}^2\,\alpha - \cot^2\alpha)^2 + 3\,\text{cosec}^2\,\alpha\cot^2\alpha$$
$$= 1 + 3\,\text{cosec}^2\,\alpha\cot^2\alpha = \text{LHS}.$$

Example 1.86: If $A = 580°$, prove that:

$$2\sin\frac{A}{2} = -\sqrt{1+\sin A} - \sqrt{1-\sin A}\,.$$

Solution: $A = 580° \Rightarrow \frac{A}{2} = 290°$

Now, $\sin\frac{A}{2} + \cos\frac{A}{2} = \sin 290° + \cos 290°$

$$= \sin(270° + 20°) + \cos(270° + 20°)$$
$$= -\cos 20° + \sin 20° = -\text{ ve (as } \cos 20° > \sin 20°)$$

Again, $\sin\frac{A}{2} - \cos\frac{A}{2} = -\cos 20° - \sin 20° = -\text{ ve}$

Now, $\left[\sin\frac{A}{2} + \cos\frac{A}{2}\right]^2 = 1 + \sin A \;\Rightarrow\; \sin\frac{A}{2} + \cos\frac{A}{2} = -\sqrt{1+\sin A}$

Similarly, $\left[\sin\frac{A}{2} - \cos\frac{A}{2}\right] = -\sqrt{1-\sin A}$

Therefore, $2\sin\frac{A}{2} = -\sqrt{1+\sin A} - \sqrt{1-\sin A}$

This proves the result.

Example 1.87: Prove that $\tan A\tan(60° + A)\tan(120° + A) = -\tan 3A$.

Solution: LHS $= \tan A\tan(60° + A)\tan(120° + A)$

$$= \tan A\left(\frac{\tan 60° + \tan A}{1 - \tan A\tan 60°}\right)\left(\frac{\tan 120° + \tan A}{1 - \tan A\tan 120°}\right)$$
$$= \tan A\left(\frac{\sqrt{3} + \tan A}{1 - \sqrt{3}\tan A}\right)\left(\frac{-\sqrt{3} + \tan A}{1 + \sqrt{3}\tan A}\right)$$
$$= \tan A\left(\frac{\tan^2 A - 3}{1 - 3\tan^2 A}\right) = \frac{\tan^3 A - 3\tan A}{1 - 3\tan^2 A}$$
$$= -\left(\frac{3\tan A - \tan^3 A}{1 - 3\tan^2 A}\right) = -\tan 3A.$$

NOTES

Example 1.88: Express $\sqrt{3}\cos\theta + \sin\theta$ as cosine of an angle. Hence, Find their greatest and least values.

Solution: $\sqrt{3}\cos\theta + \sin\theta$

$$= 2\left(\frac{\sqrt{3}}{2}\cos\theta + \frac{1}{2}\sin\theta\right)$$

$$= 2\left(\cos\frac{\pi}{6}\cos\theta + \sin\frac{\pi}{6}\sin\theta\right) = 2\cos\left(\frac{\pi}{6} - \theta\right) \text{ or } 2\cos\left(\theta - \frac{\pi}{6}\right).$$

Since the greatest value of $\cos\alpha = 1$ and least value is -1, the greatest and least values of $\sqrt{3}\cos\theta + \sin\theta$ are 2 and -2 respectively.

Example 1.89: If $A + B = \frac{\pi}{4}$, prove that $(1 + \tan A)(1 + \tan B) = 2$.

Solution: LHS $= (1 + \tan A)(1 + \tan B)$

$= 1 + \tan A + \tan B + \tan A \tan B$

Now, $\tan(A + B) = \tan\frac{\pi}{4}$

$$\Rightarrow \frac{\tan A + \tan B}{1 - \tan A \tan B} = 1$$

$\Rightarrow \tan A + \tan B = 1 - \tan A \tan B$

Therefore, LHS $= 1 + 1 - \tan A \tan B + \tan A \tan B = 2 =$ RHS.

Example 1.90: Prove that:

$$\frac{1 - \sin 36° + \cos 36°}{1 + \sin 36° + \cos 36°} = \frac{3\tan 9° - \tan^3 9°}{1 - 3\tan^2 9°}$$

Solution: LHS$= \dfrac{(2\cos^2 18°) - \sin 36°}{(2\cos^2 18°) + \sin 36°}$

$$= \frac{2\cos^2 18° - 2\sin 18°\cos 18°}{2\cos^2 18° + 2\sin 18°\cos 18°}$$

$$= \frac{2\cos 18°(\cos 18° - \sin 18°)}{2\cos 18°(\cos 18° + \sin 18°)}$$

$$= \frac{1 - \tan 18°}{1 + \tan 18°} = \tan\left(\frac{\pi}{4} - 18°\right)$$

$= \tan(45° - 18°) = \tan(27°)$

RHS $= \tan 3(9°) = \tan 27°$.

Thus, LHS $=$ RHS.

Example 1.91: Prove that

$$\cos^4\frac{\pi}{8} + \cos^4\frac{3\pi}{8} + \cos^4\frac{5\pi}{8} + \cos^4\frac{7\pi}{8} = \frac{3}{2}.$$

Solution:

$$\text{LHS} = \cos^4 22\frac{1}{2}^\circ + \cos^4 67\frac{1}{2}^\circ + \cos^4 112\frac{1}{2}^\circ + \cos^4 157\frac{1}{2}^\circ$$

$$= \left(\frac{1+\cos 45°}{2}\right)^2 + \left(\frac{1+\cos 135°}{2}\right)^2 + \left(\frac{1+\cos 225°}{2}\right)^2 + \left(\frac{1+\cos 315°}{2}\right)^2$$

NOTES

$$= \left(\frac{1+\frac{1}{\sqrt{2}}}{2}\right)^2 + \left(\frac{1-\frac{1}{\sqrt{2}}}{2}\right)^2 + \left(\frac{1-\frac{1}{\sqrt{2}}}{2}\right)^2 + \left(\frac{1+\frac{1}{\sqrt{2}}}{2}\right)^2$$

$$= \frac{1+\frac{1}{2}+\sqrt{2}+1+\frac{1}{2}-\sqrt{2}}{2} = \frac{3}{2} = \text{RHS.}$$

Example 1.92: Prove that $\sin\theta\tan\theta$ is greater than $2(1-\cos\theta)$, if θ is an acute angle. Indicate the step where you have used the fact that θ should be an acute angle.

Solution: Now, $\sin\theta\tan\theta - 2(1-\cos\theta)$

$$= \frac{2\tan\frac{\theta}{2}}{1+\tan^2\frac{\theta}{2}} \; \frac{2\tan\frac{\theta}{2}}{1-\tan^2\frac{\theta}{2}} - 2\left(2\sin^2\frac{\theta}{2}\right)$$

$$\left(\text{because } \sin\theta = 2\sin\frac{\theta}{2}\cos\frac{\theta}{2} = \frac{2\sin\frac{\theta}{2}\cos\frac{\theta}{2}}{\sin^2\frac{\theta}{2}+\cos^2\frac{\theta}{2}} = \frac{2\tan\frac{\theta}{2}}{1+\tan^2\frac{\theta}{2}}\right)$$

$$= \frac{4\tan^2\frac{\theta}{2}}{\sec^2\frac{\theta}{2}} \; \frac{1}{1-\tan^2\frac{\theta}{2}} - 4\sin^2\frac{\theta}{2}$$

$$= 4\left[\frac{\sin^2\frac{\theta}{2}\;\cos^2\frac{\theta}{2}}{\cos^2\frac{\theta}{2}-\sin^2\frac{\theta}{2}} - \sin^2\frac{\theta}{2}\right]$$

$$= 4\sin^2\frac{\theta}{2}\left[\frac{\cos^2\frac{\theta}{2}}{\cos^2\frac{\theta}{2}-\sin^2\frac{\theta}{2}} - 1\right]$$

$$= 4\sin^2\frac{\theta}{2}\left[\frac{\sin^2\frac{\theta}{2}}{\cos^2\frac{\theta}{2}-\sin^2\frac{\theta}{2}}\right]$$

$$= 4\sin^2\frac{\theta}{2}\left[\frac{\sin^2\frac{\theta}{2}}{\cos\theta}\right] > 0 \qquad [\text{Since } \theta \text{ is acute angle, } \cos\theta > 0]$$

$\Rightarrow \sin\theta\tan\theta > 2(1-\cos\theta)$.

NOTES

Transformation of Products into Sums or Differences

We know that

$$\sin A \cos B + \sin B \cos A = \sin (A + B) \quad ...(1.24)$$

and $$\sin A \cos B - \sin B \cos A = \sin (A - B) \quad ...(1.25)$$

Adding equation (1.24) and (1.25), we get

$$2 \sin A \cos B = \sin (A + B) + \sin (A - B)$$

Subtracting equation (1.24) and (1.25), we get

$$2 \cos A \sin B = \sin (A + B) - \sin (A - B)$$

Again,

$$\cos A \cos B + \sin A \sin B = \cos (A - B) \quad ...(1.26)$$

$$\cos A \cos B - \sin A \sin B = \cos (A + B) \quad ...(1.27)$$

Adding equation (1.26) and (1.27), we get

$$2 \cos A \cos B = \cos (A - B) + \cos (A + B)$$

Subtracting equation (1.26) and (1.27), we get

$$2 \sin A \sin B = \cos (A - B) - \cos (A + B)$$

Thus, we have the following formulas:

$$\mathbf{2 \sin A \cos B = \sin (A + B) + \sin (A - B)}$$
$$\mathbf{2 \cos A \sin B = \sin (A + B) - \sin (A - B)}$$
$$\mathbf{2 \cos A \cos B = \cos (A + B) + \cos (A - B)}$$
$$\mathbf{2 \sin A \sin B = \cos (A - B) - \cos (A + B)}$$

Transformation of Sums or Differences into Products

To prove that for all angles C and D,

$$\mathbf{\sin C + \sin D = 2 \sin \frac{C + D}{2} \cos \frac{C - D}{2}}$$

$$\mathbf{\sin C - \sin D = 2 \cos \frac{C + D}{2} \sin \frac{C - D}{2}}$$

$$\mathbf{\cos C + \cos D = 2 \cos \frac{C + D}{2} \cos \frac{C - D}{2}}$$

$$\mathbf{\cos C - \cos D = 2 \sin \frac{C + D}{2} \sin \frac{D - C}{2}}$$

Proof: We know that for all values of A and B,

$$\sin (A + B) = \sin A \cos B + \cos A \sin B$$

$$\sin (A - B) = \sin A \cos B - \cos A \sin B$$

By addition and subtraction, we have

$$\sin (A + B) + \sin (A - B) = 2 \sin A \cos B$$

$$\sin (A + B) - \sin (A - B) = 2 \sin A \cos B.$$

Put $A + B = C$ and $A - B = D$

NOTES

We get $A = \frac{C+D}{2}$, $B = \frac{C-D}{2}$.

Then, $\sin C + \sin D = 2\sin\frac{C+D}{2}\cos\frac{C-D}{2}$

$\sin C - \sin D = 2\cos\frac{C+D}{2}\sin\frac{C-D}{2}$.

Similarly, by adding and subtracting the relations,

$\cos(A+B) = \cos A\cos B - \sin A\sin B$

$\cos(A-B) = \cos A\cos B + \sin A\sin B$.

We have,

$\cos(A+B) + \cos(A-B) = 2\cos A\cos B$

$\cos(A+B) - \cos(A-B) = -2\sin A\sin B$.

Put $A + B = C$, $A - B = D$

So that $A = \frac{C+D}{2}$, $B = \frac{C-D}{2}$.

Hence, $\cos C + \cos D = 2\cos\frac{C+D}{2}\cos\frac{C-D}{2}$

$\cos C - \cos D = -2\sin\frac{C+D}{2}\sin\frac{C-D}{2}$

$= 2\sin\frac{C+D}{2}\sin\frac{D-C}{2}$.

To prove that,

(i) $\sin(\mathrm{A}+\mathrm{B})\ \sin(\mathrm{A}-\mathrm{B}) = \sin^2\mathrm{A} - \sin^2\mathrm{B}$

(ii) $\cos(\mathrm{A}+\mathrm{B})\ \cos(\mathrm{A}-\mathrm{B}) = \cos^2\mathrm{A} - \sin^2\mathrm{B} = \cos^2\mathrm{B} - \sin^2\mathrm{A}$

Proof: (i) $\sin(A+B)\sin(A-B)$

$= (\sin A\cos B + \cos A\sin B)(\sin A\cos B - \cos A\sin B)$

$= \sin^2 A\cos^2 B - \cos^2 A\sin^2 B$

$= \sin^2 A(1 - \sin^2 B) - (1 - \sin^2 A)\sin^2 B$

$= \sin^2 A - \sin^2 A\sin^2 B - \sin^2 B + \sin^2 A\sin^2 B$

$= \sin^2 A - \sin^2 B$.

(ii) $\cos(A+B)\cos(A-B)$.

$= (\cos A\cos B - \sin A\sin B)(\cos A\cos B + \sin A\sin B)$

$= \cos^2 A\cos^2 B - \sin^2 A\sin^2 B$

$= \cos^2 A(1 - \sin^2 B) - (1 - \cos^2 A)\sin^2 B$

$= \cos^2 A - \cos^2 A\sin^2 B - \sin^2 B + \cos^2 A\sin^2 B$

$= \cos^2 A - \sin^2 B$

$= (1 - \sin^2 A) - (1 - \cos^2 B)$

$= \cos^2 B - \sin^2 A$

NOTES

Example 1.93: If the angles A, B, C, of a triangle are in A.P, show that:

$$\frac{\cos C - \cos A}{\sin A - \sin C} = \tan B.$$

Solution: Since in any triangle

$$A + B + C = \pi$$

and $\quad C = \pi-(A + B)$

$$\cos C = -\cos (A + B)$$

and $\quad \sin C = \sin (A + B)$

$$\text{LHS} = \frac{-\cos (A + B) - \cos A}{\sin A - \sin (A + B)}$$

$$= \frac{-2\cos\left(A + \frac{B}{2}\right)\cos\frac{B}{2}}{-2\cos\left(A + \frac{B}{2}\right)\sin\frac{B}{2}}$$

$$= \cot\frac{B}{2} = \cot\left(\frac{\pi}{2} - \left[\frac{A+C}{2}\right]\right)$$

$$= \tan\left[\frac{A+C}{2}\right] = \tan B$$

$$\left(\text{As } A, B, C \text{ are in A.P.} \Rightarrow \frac{A+C}{2} = B\right)$$

Example 1.94 Prove that:

$$\frac{\sin^2 A - \sin^2 B}{\sin A \cos A - \sin B \cos B} = \tan (A + B).$$

Solution: $\text{LHS} = \dfrac{2\sin (A + B) \sin (A - B)}{\sin^2 A - \sin^2 B}$

$$= \frac{2\sin (A + B) \sin (A - B)}{2\cos (A + B) \sin (A - B)} = \tan (A + B) = \text{RHS}.$$

Example 1.95: Prove that $4(\cos^3 10° + \sin^3 20°) = 3(\cos 10° + \sin 20°)$.

Solution: $\text{LHS} = 4(\cos 10° + \sin 20°)\ (\cos^2 10° + \sin^2 20° + \cos 10° \sin 20°)$

$$= 4(\cos 10° + \sin 20°)\ (1 - \sin^2 10° + \sin^2 20° + \cos 10° \sin 20°)$$

$$= 4(\cos 10° + \sin 20°)\ (1 + \sin 30° \sin 10° - \cos 10° \sin 20°)$$

$$= 4(\cos 10° + \sin 20°)\left[1 + \frac{\sin 10°}{2} - \frac{(\sin 30° + \sin 10°)}{2}\right].$$

$$= 2(\cos 10° + \sin 20°)\left[2 + \sin 10° - \frac{1}{2} - \sin 10°\right]$$

$$= 2(\cos 10° + \sin 20°)\left[\frac{3}{2}\right]$$

$$= 3(\cos 10° + \sin 20°) = \text{RHS}.$$

NOTES

Example 1.96: Given $x = \cos 55°$, $y = \cos 65°$
$z = \cos 175°$. Prove that:

$$x + y + z = 0 \text{ and } xy + yz + zx = -\frac{3}{4}.$$

Solution: $x + y + z = \cos 55° + \cos 65° + \cos 175°$

$$= 2 \cos 60° \cos 5° + \cos (180° - 5°)$$

$$= \cos 5° - \cos 5° = 0.$$

$$xy + yz + zx = \cos 55° \cos 65° + \cos 65° \cos 175° + \cos 175° \cos 55°$$

$$= \frac{1}{2} (\cos 120° + \cos 10°) + \frac{1}{2} (\cos 240° + \cos 110°)$$

$$+ \frac{1}{2} (\cos 230° + \cos 120°)$$

$$= -\frac{1}{4} + \frac{1}{2} \cos 10° - \frac{1}{4} + \frac{1}{2} \cos 110°$$

$$+ \frac{1}{2} \cos 230° - \frac{1}{4}$$

$$= -\frac{3}{4} + \frac{1}{2} [\cos 10° + \cos 110° + \cos 230°]$$

$$= -\frac{3}{4} + \frac{1}{2} [2 \cos 60° \cos 50° - \cos 50°]$$

$$= -\frac{3}{4} + \frac{1}{2} [\cos 50° - \cos 50°] = -\frac{3}{4}.$$

Example 1.97: Prove that:

$$\sin^2 B = \sin^2 A + \sin^2 (A - B) - 2 \sin A \cos B \sin (A - B).$$

Solution: RHS$= \sin^2 A + \sin^2 (A - \text{B}) - (2 \sin A \cos B) \sin (A - B)$

$$= \sin^2 A + \sin^2 (A - \text{B}) - [\sin (A + B)$$

$$+ \sin (A - B)] \times \sin (A - B)$$

$$= \sin^2 A + \sin^2 (A - \text{B}) - \sin (A + B)$$

$$\sin (A - B) - \sin^2 (A - B)$$

$$= \sin^2 A - \sin (A + \text{B}) \sin (A - B)$$

$$= \sin^2 A - (\sin^2 A - \sin^2 B)$$

$$= \sin^2 B = \text{LHS}.$$

Example 1.98: If $y \sin \alpha = x \sin (2\beta + \alpha)$
show that $(x + y) \cot (\alpha + \beta) = (y - x) \cot \beta$.

Solution: $y \sin \alpha = x \sin (2\beta + \alpha)$

$$\Rightarrow \quad \frac{y}{x} = \frac{\sin (2\beta + \alpha)}{\sin \alpha}$$

Applying componendo and dividendo

NOTES

$$\Rightarrow \quad \frac{y+x}{y-x} = \frac{\sin(2\beta+\alpha)+\sin\alpha}{\sin(2\beta+\alpha)-\sin\alpha}$$

$$\Rightarrow \quad \frac{y+x}{y-x} = \frac{2\sin(\alpha+\beta)\cos\beta}{2\cos(\alpha+\beta)\sin\beta} \quad \Rightarrow \quad \frac{y+x}{y-x} = \tan(\alpha+\beta)\cot\beta$$

$$\Rightarrow \quad (y+x)\cot(\alpha+\beta) = (y-x)\cot\beta$$

This proves the result.

Example 1.99: Prove that:

$$\left[\frac{\cos A+\cos B}{\sin A-\sin B}\right]^n + \left[\frac{\sin A+\sin B}{\cos A-\cos B}\right]^n$$

$$= 2\cot^n\left[\frac{A-B}{2}\right] \text{ or 0 according as } n \text{ is even or odd.}$$

Solution: $\left[\frac{\cos A+\cos B}{\sin A-\sin B}\right]^n + \left[\frac{\sin A+\sin B}{\cos A-\cos B}\right]^n$

$$= \left[\frac{2\cos\frac{A+B}{2}\cos\frac{A-B}{2}}{2\sin\frac{A-B}{2}\cos\frac{A+B}{2}}\right]^n + \left[\frac{2\sin\frac{A+B}{2}\cos\frac{A-B}{2}}{2\sin\frac{A+B}{2}\sin\frac{B-A}{2}}\right]^n$$

$$= \left[\cot\frac{(A-B)}{2}\right]^n + \left[-\cot\frac{(A-B)}{2}\right]^n$$

So, LHS = $2\cot^n\left[\frac{A-B}{2}\right]$ if n is even and 0 if n is odd.

Example 1.100: If $\sin x + \sin y = a$ and $\cos x + \cos y = b$ show that

$$\cos(x+y) = \frac{b^2-a^2}{b^2+a^2}$$

Solution: Now, $\sin x + \sin y = a$

$$\Rightarrow \quad 2\sin\frac{x+y}{2}\cos\frac{x-y}{2} = a$$

and $\quad \cos x + \cos y = b$

$$\Rightarrow \quad 2\cos\frac{x+y}{2}\cos\frac{x-y}{2} = b$$

Therefore, $\tan\left[\frac{x+y}{2}\right] = \frac{a}{b}$

Now, $\cos(x+y) = \dfrac{1-\tan^2\left[\frac{x+y}{2}\right]}{1+\tan^2\left[\frac{x+y}{2}\right]} = \dfrac{1-\frac{a^2}{b^2}}{1+\frac{a^2}{b^2}} = \dfrac{b^2-a^2}{b^2+a^2}.$

Example 1.101: Prove that in a triangle ABC:

$$\frac{\sin(A-B)}{\sin A\sin B} + \frac{\sin(B-C)}{\sin B\sin C} + \frac{\sin(C-A)}{\sin C\sin A} = 0.$$

NOTES

Solution: LHS $= \dfrac{\sin C \sin(A-B) + \sin A \sin(B-C) + \sin B \sin(C-A)}{\sin A \sin B \sin C}$

$$= \frac{\sin(A+B)\sin(A-B) + \sin(B+C)\sin(B-C) + \sin(C+A)\sin(C-A)}{\sin A \sin B \sin C}$$

(Using $A + B + C = \pi$)

$$= \frac{(\sin^2 A - \sin^2 B) + (\sin^2 B - \sin^2 C) + (\sin^2 C - \sin^2 A)}{\sin A \sin B \sin C}$$

$= 0 =$ RHS.

Example 1.102: Prove that, $1 + \cos 56° + \cos 58° - \cos 66°$
$= 4 \cos 28° \cos 29° \sin 33°$

Solution: Now, $1 - \cos 66° = 2 \sin^2 33°$.

Therefore, LHS $= 2 \sin^2 33° + (\cos 56° + \cos 58°)$
$= 2 \sin^2 33° + 2 \cos 57° \cos 1°$
$= 2 \sin^2 33° + 2 \cos (90° - 33°) \cos 1°$
$= 2 \sin^2 33° + 2 \sin 33° \cos 1°$
$= 2 \sin 33° (\sin 33° + \cos 1°)$
$= 2 \sin 33° (\cos 57° + \cos 1°)$
$= 2 \sin 33° (2 \cos 29° \cos 28°)$
$= 4 \cos 28° \cos 29° \sin 33° =$ RHS.

Example 1.103: Prove that $\sin^2 12° + \sin^2 21° + \sin^2 39° + \sin^2 48°$
$= 1 + \sin^2 9° + \sin^2 18°$.

Solution: Consider

$(\sin^2 12° - \sin^2 18°) + (\sin^2 21° - \sin 9°)$
$= (-\sin 30° \sin 6°) + (\sin 30° \sin 12°)$
$= \sin 30° (\sin 12° - \sin 6°)$
$= \dfrac{1}{2} (2 \sin 3° \cos 9°) = \sin 3° \cos 9°$

Also, $1 - \sin^2 39° - \sin^2 48° = \cos^2 39° - \sin^2 48°$
$= \cos 87° \cos 9° = \sin 3° \cos 9°$

Therefore, $(\sin^2 12° - \sin^2 18°) + (\sin^2 21° - \sin^2 9°)$
$= 1 - \sin^2 39° - \sin^2 48°$.

The proves the result.

Example 1.104: Prove that $\cos 10° \cos 50° \cos 70° = \dfrac{\sqrt{3}}{8}$.

Solution: LHS $= \cos 10° \cos 50° \cos 70°$
$= \cos 10° \left[\dfrac{1}{2}(\cos 120° + \cos 20°)\right]$

NOTES

$$= \frac{1}{2}\cos 10° \left(-\frac{1}{2} + \cos 20°\right)$$

$$= \frac{1}{2}\cos 10° \left(-\frac{1}{2} + 2\cos^2 10° - 1\right)$$

$$= \frac{1}{2}\cos 10° \left(-\frac{3}{2} + 2\cos^2 10°\right)$$

$$= \frac{1}{4}\cos 10° \,(4\cos^2 10° - 3)$$

$$= \frac{1}{4}(4\cos^3 10° - 3\cos 10°)$$

$$= \frac{1}{4}\cos 3 \times 10° = \frac{1}{4}\cos 30°$$

$$= \frac{1}{4}\frac{\sqrt{3}}{2} = \frac{\sqrt{3}}{8} = \text{RHS.}$$

Example 1.105: If $A + B + C = \pi$, then prove that:
$\tan A + \tan B + \tan C = \tan A \tan B \tan C$.

Solution: $A + B + C = \pi$

$\Rightarrow \quad A + B = \pi - C$

$\Rightarrow \quad \tan(A + B) = -\tan C$

$\Rightarrow \quad \dfrac{\tan A + \tan B}{1 - \tan A \tan B} = -\tan C$

$\Rightarrow \tan A + \tan B + \tan C = \tan A \tan B \tan C$

This proves the result.

Example 1.106: If $A + B + C = \pi$, then prove that:

$$\sin^2\frac{A}{2} + \sin^2\frac{B}{2} + \sin^2\frac{C}{2} = 1 - 2\, sin\frac{A}{2}\, sin\frac{B}{2}\, sin\frac{C}{2}.$$

Solution: Consider,

$$1 - \sin^2\frac{A}{2} - \sin^2\frac{B}{2} - \sin^2\frac{C}{2} = \cos^2\frac{A}{2} - \sin^2\frac{B}{2} - \sin^2\frac{C}{2}$$

$$= \cos\left[\frac{A+B}{2}\right]\cos\left[\frac{A-B}{2}\right] - \sin^2\frac{C}{2}$$

$$= \cos\left[\frac{A+B}{2}\right]\cos\left[\frac{A-B}{2}\right] - \cos^2\left(\frac{A+B}{2}\right) \text{ as } \frac{C}{2}$$

$$= \frac{\pi}{2} - \left(\frac{A+B}{2}\right)$$

$$= \cos\left(\frac{A+B}{2}\right)\left[\cos\left(\frac{A-B}{2}\right) - \cos\left(\frac{A+B}{2}\right)\right]$$

$$= \cos\left(\frac{A+B}{2}\right)\left[2\sin\frac{A}{2}\sin\frac{B}{2}\right]$$

$$= 2\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}$$

Hence, $\sin^2 \frac{A}{2} + \sin^2 \frac{B}{2} + \sin^2 \frac{C}{2} = 1 - 2 \sin\frac{A}{2} \sin\frac{B}{2} \sin\frac{C}{2}$.

NOTES

Example 1.107: If $A + B + C = \pi$, prove that:

$$\cos^2 A + \cos^2 B + \cos^2 C = 1 - 2 \cos A \cos B \cos C.$$

Solution: Consider $1 - (\cos^2 A + \cos^2 B + \cos^2 C)$

$$= (1 - \cos^2 A) - \cos^2 B - \cos^2 C$$
$$= \sin^2 A - \cos^2 B - \cos^2 C$$
$$= - \cos (A + B) \cos (A - B) - \cos^2 C$$
$$= - \cos (\pi - C) \cos (A - B) - \cos^2 C$$
$$= \cos C \cos (A - B) - \cos^2 C$$
$$= \cos C [\cos (A - B) - \cos C]$$
$$= \cos C [\cos (A - B) - \cos (\pi - (A + B)]$$
$$= \cos C [\cos (A - B) + \cos (A + B)]$$
$$= \cos C [2 \cos A \cos B]$$
$$= 2 \cos A \cos B \cos C$$

Thus, $\cos^2 A + \cos^2 B + \cos^2 C = 1 - 2\cos A \cos B \cos C$

This proves the result.

CHECK YOUR PROGRESS

17. Prove that $\sin 105° + \cos 105° = \cos 45°$.
18. Find the value of $\tan 75°$ and hence prove that $\tan 75° + \cot 75° = 4$.
19. Prove that $\dfrac{\cos 13° + \sin 13°}{\cos 13° - \sin 13°} = \tan 58°$.
20. Prove that $\cot 22\frac{1}{2}^{\circ} - \tan 22\frac{1}{2}^{\circ} = 2$
21. Prove that $\dfrac{\cos A + \sin A}{\sqrt{1 + \sin 2A}} = 1$.
22. If, $A + B + C = \pi$, show that
    $\sin 2A + \sin 2B - \sin 2C = 4 \cos A \cos B \sin C$.
23. In a $\Delta\, ABC$, prove that
    $\sin 2A + \sin 2B + \sin 2C = 4 \sin A \sin B \sin C$.

1.9 SUMMARY

In this unit, you have learned that:

- Trigonometry is that branch of mathematics which deals with the measurement of angles.

NOTES

- The term 'trigonometry' is derived from two Greek words '*trigonon*' (a triangle) and '*metron*' (a measure), meaning the measurement of triangles.
- Nowadays, the term trigonometry is used for the measurement of angles in general, whether the angles are of a triangle or not.
- An angle is defined as the rotation of a line about one of its extremities in a plane from one position to another.
- Two lines are said to be at right angles, if a revolving line starting from one position to another describes one-quarter of a circle.
- When a revolving line moves in the anticlockwise direction, the angle described by it is said to be positive; otherwise, it is called negative.
- To measure angles, a particular angle is fixed and is taken as a unit of measurement so that any other angle is measured by the number of times it contains that unit.
- There are three systems of measurement:

  (*i*) Sexagesimal system: In this system, a right angle is divided into 90 equal parts called degrees. Each degree is divided into 60 equal parts called minutes and each minute is further subdivided into 60 equal parts called seconds.

  $$\text{Thus,}\quad \begin{aligned} 1 \text{ right angle} &= 90 \text{ degrees} \\ 1 \text{ degree} &= 60 \text{ minutes} \\ 1 \text{ minute} &= 60 \text{ seconds} \end{aligned}$$

  In symbols, a degree, a minute and a second are respectively written as $1°$; $1'$, $1''$.

  (*ii*) Centesimal system: In this system, a right angle is divided into 100 equal parts called grades. Each grade is divided into 100 equal parts called minutes and each minute is divided into 100 equal parts called seconds.

  $$\text{Thus,}\quad \begin{aligned} 1 \text{ right angle} &= 100 \text{ grades} \\ 1 \text{ grade} &= 100 \text{ minutes} \\ 1 \text{ minute} &= 100 \text{ seconds} \end{aligned}$$

  (*iii*) Circular system: In this system, the unit of measurement is radian. A radian is defined as the angle subtended at the centre of a circle by an arc equal to the radius of the circle.
- Trigonometry deals with the problem of measurement of triangles and periodic functions.
- The applications of trigonometry to business cycles and other situations are concerned with the properties and applications of circular or periodic functions.
- A function with period $p (p \neq 0)$ if $f(x + p) = f(x)$ is periodic.
- The amplitude of a sine wave is the absolute value of one half of the difference between the greatest and the least ordinates of the wave.

NOTES

1.10 KEY TERMS

- **Trigonometry:** It is that branch of mathematics which deals with measurement of angles.
- **Angle:** It is defined as the rotation of a line about one of its extremities in a plane from one position to another.
- **Radian:** It is the angle subtended at the centre of a circle by an Arc equal to the radius of the circle.

1.11 ANSWERS TO 'CHECK YOUR PROGRESS'

1. In case the revolving line moves in anticlockwise direction, then the angle described by it is said to be positive, else, it is called negative.
2. A radian is the measure of the angle made at the centre of a circle by an arc whose length equals the radius of the circle.
3. The area of a circle with radius r is πr^2.
4. The area of a sector AOB subtending an angle θ at the centre is $\frac{1}{2}r^2\theta$.
5. (*i*) If a line is horizontal or parallel to the x-axis, its inclination is zero, i.e., $\theta = 0$.

   (*ii*) If a line is perpendicular to the x-axis, $\theta = 90°$.
6. The fundamental period of a periodic function like $f(t) = \sin bt$ or $g(t) = \cos bt$ is given by:

$$T = \frac{2\pi}{|b|}$$

7. If $\cot\theta$ is positive, θ lies in first or third quadrant.

   If $\operatorname{cosec}\theta$ is negative, θ lies in third or fourth quadrant.

   In order that $\cot\theta$ is positive and $\operatorname{cosec}\theta$ is negatie, we see that θ must lie in third quadrant.
8. $$\frac{2\sin\theta + 3\cos\theta}{4\cos\theta + 3\sin\theta} = \frac{2\tan\theta + 3}{4 + 3\tan\theta} = \frac{\frac{8}{5} + 3}{4 + \frac{12}{5}} = \frac{23}{32}.$$
9. $$\frac{p\cos\theta + q\sin\theta}{p\cos\theta - q\sin\theta} = \frac{p\frac{\cos\theta}{\sin\theta} + q}{p\frac{\cos\theta}{\sin\theta} - q} = \frac{p\cot\theta + q}{p\cot\theta - q} = \frac{\frac{p^2}{q} + q}{\frac{p^2}{q} - q} = \frac{p^2 + q^2}{p^2 - q^2}.$$

NOTES

10. $\tan\theta = 2\sin\theta \Rightarrow \sin\theta = 2\sin\theta\cos\theta$

$$\Rightarrow \sin\theta\,(1 - 2\cos\theta) = 0$$

$$\Rightarrow \sin\theta = 0 \quad \text{or} \quad 1 - 2\cos\theta = 0$$

$\sin\theta = 0 \Rightarrow \theta = 0^\circ$ as θ is acute.

$1 - 2\cos\theta = 0 \Rightarrow \cos\theta = \frac{1}{2} \Rightarrow \theta = 60^\circ = \pi/3.$

This angle is also acute. So, $\theta = 0$ or $\pi/3$

11. Since $A + B + C = 180^\circ$

$$A + B = 180^\circ - C.$$

$$\Rightarrow \cot(A + B) + \cot C = \cot(180^\circ - C) + \cot C$$

$$= -\cot C + \cot C = 0.$$

12. Solving given equations for $\sin A$ and $\sin B$, we get $\sin A = \frac{1}{2}$ and $\sin B = \frac{\sqrt{3}}{2}$

$\Rightarrow \quad A = 30^\circ$ or 150°, $B = 60^\circ$ or 120°

$\Rightarrow \quad A = 30^\circ$ and $B = 60^\circ$ or 120°

If $A = 30^\circ$ and $B = 60^\circ$, then $C = 90^\circ$ and if $A = 30^\circ$ and $B = 120^\circ$, then $C = 30^\circ$.

13. $3\cos^2\theta - \sin^2\theta = 1$

$$\Rightarrow 3\cos^2\theta - (1 - \cos^2\theta) = 1$$

$$\Rightarrow 4\cos^2\theta = 2$$

$$\Rightarrow \cos^2\theta = \frac{1}{2}$$

$$\Rightarrow \cos\theta = \pm\frac{1}{\sqrt{2}}$$

Since $180^\circ \le \theta \le 270^\circ \Rightarrow \cos\theta = \frac{-1}{\sqrt{2}} \Rightarrow \theta = 225^\circ.$

14. (*i*) If $\tan^{-1}\frac{3}{4} = x$, $\tan x = \frac{3}{4}$, $\cos x = \frac{4}{5}$ $\therefore$ $x = \cos^{-1}\frac{4}{5}$

$$\cos(\tan^{-1} 3/4) = \cos\left(\cos^{-1}\frac{4}{5}\right) = \frac{4}{5}.$$

(*ii*) If $\cot^{-1} x = y$ then $\cot y = x$

$$\therefore \quad \sin y = \frac{1}{\sqrt{1+x^2}} \Rightarrow \sin\cot^{-1}x = \frac{1}{\sqrt{1+x^2}}$$

15. (*i*) $\theta = 2n\pi \pm \frac{\pi}{3}$, $\alpha = \frac{\pi}{3}$

(*ii*) $\theta = n\pi + (-1)^n\left(-\frac{\pi}{4}\right)$, $\alpha = -\frac{\pi}{4}$

(*iii*) $\theta = n\pi - \frac{\pi}{4}$, $\alpha = -\frac{\pi}{4}$ $0 \le \phi \le \pi$

16. (*i*) $\begin{cases} \theta = \pi/2 \\ \theta = \pi/4, 3\pi/4 \\ \theta = \pi/6, 3\pi/6, 5\pi/6 \end{cases}$

(*ii*) $\theta = n\pi + \pi/10$

(*iii*) $\theta = n\pi - \pi/4$

(*iv*) $\theta = n\pi \pm \dfrac{\pi}{3}$

NOTES

17.
$$\begin{aligned} \text{LHS} &= \sin 105° + \cos 105° \\ &= \sin(60° + 45°) + \cos(60° + 45°) \\ &= (\sin 60° \cos 45° + \cos 60° \sin 45°) \\ &\qquad + (\cos 60° \cos 45° - \sin 60° \sin 45°) \\ &= \left(\frac{\sqrt{3}}{2}\frac{1}{\sqrt{2}} + \frac{1}{2}\frac{1}{\sqrt{2}}\right) + \left(\frac{1}{2}\frac{1}{\sqrt{2}} - \frac{\sqrt{3}}{2}\frac{1}{\sqrt{2}}\right) = \frac{1}{\sqrt{2}} \end{aligned}$$

$$\text{RHS} = \cos 45° = \frac{1}{\sqrt{2}}.$$

Hence, the result follows.

18.
$$\begin{aligned} \tan 75° &= \tan(45° + 30°) \\ &= \frac{\tan 45° + \tan 30°}{1 - \tan 45° \tan 30°} = \frac{1 + \frac{1}{\sqrt{3}}}{1 - \frac{1}{\sqrt{3}}} \\ &= \frac{\sqrt{3+1}}{\sqrt{3-1}} = \frac{\left(\sqrt{3+1}\right)^2}{2} \\ &= \frac{4 + 2\sqrt{3}}{2} = 2 + \sqrt{3} \end{aligned}$$

$$\begin{aligned} \text{Now } \tan 75° + \cot 75° &= 2 + \sqrt{3} + \frac{1}{2 + \sqrt{3}} \\ &= (2 + \sqrt{3}) + \frac{(2 - \sqrt{3})}{(2 + \sqrt{3})(2 - \sqrt{3})} \\ &= (2 + \sqrt{3}) + \frac{(2 - \sqrt{3})}{4 - 3} \\ &= 2 + \sqrt{3} + (2 - \sqrt{3}) = 4. \end{aligned}$$

19.
$$\begin{aligned} \text{LHS} &= \frac{\cos 13° + \sin 13°}{\cos 13° - \sin 13°} \\ &= \frac{1 + \frac{\sin 13°}{\cos 13°}}{1 - \frac{\sin 13°}{\cos 13°}} = \frac{1 + \tan 13°}{1 - \tan 13°} \end{aligned}$$

$$= \frac{\tan 45° + \tan 13°}{1 - \tan 45° \tan 13°} \qquad \text{(as } \tan 45° = 1\text{)}$$

$$= \tan (45° + 13°) = \tan 58° = \text{RHS}.$$

NOTES

20. $$\text{LHS} = \cot 22\frac{1}{2}^\circ - \tan 22\frac{1}{2}^\circ$$

$$= \frac{\cos^2\left(22\frac{1}{2}\right)^\circ - \sin^2\left(22\frac{1}{2}\right)^\circ}{\sin 22\frac{1}{2}^\circ \cos 22\frac{1}{2}^\circ}$$

$$= \frac{2\cos 45°}{2\sin 22\frac{1}{2}^\circ \cos 22\frac{1}{2}^\circ} = \frac{2\cos 45°}{\sin 45°}$$

$$= 2\cot 45° = 2 = \text{RHS}.$$

21. $$\text{LHS} = \frac{\cos A + \sin A}{\sqrt{1 + \sin 2A}}$$

$$= \frac{\cos A + \sin A}{\sqrt{\cos^2 A + \sin^2 A + 2\sin A\cos A}}$$

$$= \frac{\cos A + \sin A}{\sqrt{(\cos A + \sin A)^2}} = \frac{\cos A + \sin A}{\cos A + \sin A} = 1.$$

22. $$\text{LHS} = 2\sin(A+B)\cos(A-B) - \sin 2C$$

$$= 2\sin C\cos(A-B) - 2\sin C\cos C$$

$$= 2\sin C[\cos(A-B) - \cos C]$$

$$= 2\sin C[\cos(A-B) + \cos(A+B)]$$

$$= 2\sin C[2\cos A\cos B] = 4\cos A\cos B\cos C.$$

23. $$\text{LHS} = \sin 2A + \sin 2B + \sin 2C$$

$$= 2\sin(A+B)\cos(A-B) + \sin 2C$$

$$= 2\sin(A+B)\cos(A-B) + \sin[2\pi - 2(A+B)]$$

as $A + B + C = \pi$

$$= 2\sin(A+B)\cos(A-B) - \sin 2(A+B)$$

$$= 2\sin(A+B)[\cos(A-B) - \cos(A+B)]$$

$$= 2\sin(A+B)[2\sin A\sin B]$$

$$= 2\sin(\pi - C)[2\sin A\sin B]$$

$$= 4\sin A\sin B\sin C = \text{RHS}.$$

1.12 QUESTIONS AND EXERCISES

NOTES

Short-Answer Questions

1. Show $\frac{(a+b+c)^2}{a^2+b^2+c^2} = \frac{\cot A/2 + \cot B/2 + \cot C/2}{\cot A + \cot B + \cot C}$
2. If the angles A, B, C are in AP, show

   $2\cos\frac{A-C}{2} = \frac{a+c}{\sqrt{a^2+c^2-ac}}$
3. If $A = 45°$, $B = 75°$, show $a + c\sqrt{2} - 2b = 0$.
4. If $\cos A = \frac{\sin B}{2\sin C}$, prove that the triangle is isosceles.
5. If $\cot A + \cot B + \cot C = \sqrt{3}$, prove that the triangle is equilateral.
6. Define the three systems of measurement of angles.

Long-Answer Questions

1. Find the values of (*i*) sin 36° and cos 36°, (*ii*) sin 15° and cos 15°.
2. If $\sin A = \frac{1}{\sqrt{10}}$, $\sin B = \frac{1}{\sqrt{5}}$, show that $A + B = 45°$.
3. Prove that $\tan A + \tan B = \frac{\sin(A+B)}{\cos A \cos B}$.
4. If $\tan\theta = \frac{a}{b}$, $\tan\phi = \frac{c}{d}$ prove that $\tan(\theta+\phi) = \frac{ad+dc}{bd-ac}$.
5. Prove that $\cos(A+B) + \sin(A-B) = 2\sin\left(\frac{\pi}{4}+A\right)\cos\left(\frac{\pi}{4}+B\right)$.
6. Express $\cos\theta - \sin\theta$ as sine or cosine of an angle and find the greatest and least values of $(\cos\theta - \sin\theta)$.
7. If $A + B = \frac{\pi}{4}$, prove that $(\cot A - 1)(\cot B - 1) = 2$.
8. Prove that $\frac{\sec 8A - 1}{\sec 4A - 1} = \frac{\tan 8A}{\tan 2A}$.
9. If $2\tan\alpha = 3\tan\beta$, show that $\tan(\alpha-\beta) = \frac{\sin 2\beta}{5 - \cos 2\beta}$.
10. If $\cos\alpha = \frac{3}{5}$ and $\cos\beta = \frac{5}{13}$, find the values of $\sin^2\frac{(\alpha-\beta)}{2}$ and $\cos^2\left(\frac{\alpha-\beta}{2}\right)$.
11. Prove that:

    (*i*) $\cos 5\theta = 16\cos^5\theta - 20\cos^3\theta + 5\cos\theta$

    (*ii*) $\cos 4\theta = 1 - 8\cos^2\theta + 8\cos^4\theta$

NOTES

12. Prove that $\tan(60° + A) + \tan(120° + A) + \tan A = 3\tan^3 A$.

13. Prove that $\dfrac{1+\sin 2\theta}{1-\sin 2\theta} = \tan^2\left(\dfrac{\pi}{4}+\theta\right)$.

14. Prove that $\dfrac{\sin 16A\cos 2A - \cos 6A\sin 12A}{\cos 4A\cos^2 A - \sin 6A\sin 8A} = \tan 4A$.

15. If $\sin\theta = \dfrac{5}{13}$ and θ lies in third quadrant, prove that the value of $2\cos\dfrac{\theta}{2} - 3\sin\dfrac{\theta}{2}$

$= -\dfrac{17}{\sqrt{26}}$

16. Prove that

(*i*) $\cot 7\frac{1}{2}° = \dfrac{1+\cos 15°}{\sin 15°}$

(*ii*) $\tan 7\frac{1}{2}° = \dfrac{1-\cos 15°}{\sin 15°}$.

17. Prove that $\cot 7\frac{1}{2}° = \sqrt{2}+\sqrt{3}+\sqrt{4}+\sqrt{6}$.

18. Prove that $\cos^3\left(A-\dfrac{2\pi}{3}\right)+\cos^3 A + \cos^3\left(A+\dfrac{2\pi}{3}\right) = \dfrac{3}{4}\cos^3 A$.

19. Prove that $\dfrac{\cos^3 x - \cos^3 x}{\cos x} + \dfrac{\sin^3 x + \sin^3 x}{\sin x} = 3$.

20. Prove that $\dfrac{1-\cos A+\cos B+\cos C}{1-\cos C+\cos A+\cos B} = \dfrac{\tan\frac{A}{2}}{\tan\frac{B}{2}}$ if $A + B + C = 180°$.

21. Prove that $\sin 3A + \sin 2A - \sin A = 4\sin A\cos\dfrac{A}{2}\cos\dfrac{3A}{2}$.

22. Prove that $16\cos\dfrac{2\pi}{15}\cos\dfrac{4\pi}{15}\cos\dfrac{8\pi}{15}\cos\dfrac{14\pi}{15} = 1$.

23. Prove that $\sin 3A = 4\sin A\sin(60° + A)\sin(60° - A)$.

24. Prove that $\dfrac{1+\sin^2\theta-\cos^2\theta}{1+\sin^2\theta+\cos^2\theta} = \tan\theta$.

25. Prove $\cot^{-1} x = \dfrac{1}{2}\pi - \tan^{-1} x$.

26. Prove

(*i*) $\sec^{-1} 2 = \dfrac{\pi}{2}$ (*ii*) $\text{cosec}^{-1}(-1) = \dfrac{3}{2}\pi$ (*iii*) $\sec^{-1}(-2) = \dfrac{4}{3}\pi$

(*iv*) $\cos^{-1}\left(\dfrac{3}{\sqrt{5}}\right)+\cos^{-1}\left(\dfrac{2}{\sqrt{5}}\right) = \dfrac{\pi}{4}$ (*v*) $2\tan^{-1}\dfrac{1}{3}\tan^{-1}\left(-\dfrac{1}{7}\right) = \dfrac{\pi}{4}$.

27. Prove that $\cos^2 A + \cos^2 B + \cos^2 C = 1 - 2\cos A \cos B \cos C$

28. Prove that $\sin^2\frac{A}{2} + \sin^2\frac{B}{2} + \sin^2\frac{C}{2} = 1 - 2\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}$

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29. Prove that $\sin(B + C - A) + \sin(C + A - B) + \sin(A + B - C) = 4\sin A \sin B \sin C$

30. Find the measure of the angle between the hour hand and the minute hand of a clock at a quarter past one.

31. Show that:

(i) $(\sin^6 A + \cos^6 A) = 3(\sin^4 A + \cos^4 A) - 1$

(ii) $\dfrac{\tan A + \sec A - 1}{\tan A - \sec A + 1} = \dfrac{1 + \sin A}{\cos A}$

(iii) $\dfrac{\cot A + \tan B}{\cot B + \tan A} = \cot A \tan B$

(iv) $(\operatorname{cosec} A - \sin A)(\sec A - \cos A)(\tan A + \cot A) = 1$

(v) $\left(\dfrac{1 + \sin\theta - \cos\theta}{1 + \sin\theta + \cos\theta}\right)^2 = \dfrac{1 - \cos\theta}{1 + \cos\theta}$

32. If tan q = $\dfrac{p}{q}$, show that $\dfrac{p\sin\theta - q\cos\theta}{p\sin\theta + q\cos\theta} = \dfrac{p^2 - q^2}{p^2 + q^2}$

Show that $\dfrac{1 + \sin A}{1 + \cos A}\;\dfrac{1 + \sec A}{1 + \operatorname{cosec} A} = \tan A$

33. Prove that:

(*i*) $\dfrac{\sec x + \operatorname{cosec} x}{1 + \tan^2 x} = \operatorname{cosec} x.$

(*ii*) $(1 + \sec\theta)(1 - \cos\theta) = \tan\theta \sin\theta$

(*iii*) $\dfrac{1 - \cot^2 x}{1 + \cot^2 x} = \sin^2 x - \cos^2 x.$

(*iv*) $(\tan x + \cos x)^2 = \sec^2 x + \operatorname{cosec}^2 x.$

(*v*) $\dfrac{\sin^3\theta + \cos^3\theta}{\sin\theta + \cos\theta} = 1 - \sin\theta\cos\theta.$

(*vi*) $\sec x + \tan x = \dfrac{1}{\sec x - \tan x}.$

(*vii*) $\dfrac{\cos x}{1 + \sin x} = \dfrac{1 - \sin x}{\cos x}.$

NOTES

(*viii*) $\sec x + \tan x = \dfrac{1}{\sec x - \tan x}$.

(*ix*) $\dfrac{\cos x}{1+\sin x} = \dfrac{1-\sin x}{\cos x}$.

(*x*) $\sin^3 t + \cos^3 t + \sin t \cos^2 t + \sin^2 \cos t = \sin t + \cos t$.

34. Prove the following identities:

(*i*) $\dfrac{\cos 2A + \cos 4A}{\sin 4A - \sin 2A} = \cot A$.

(*ii*) $\dfrac{\sin 2A - \sin 2B}{\sin 2A + \sin 2B} = \dfrac{\tan (A-B)}{\tan (A+B)}$.

(*iii*) $\sin^3 x \cos^3 x = \dfrac{3\sin 2x - \sin 6x}{32}$.

(*iv*) $\sin 2\alpha + \sin 2\beta + \sin 2\gamma = 4\cos\alpha\cos\beta\cos\gamma$, where $\alpha+\beta+\gamma=\pi$

(*v*) $\dfrac{1-\cos 8x}{8} = \sin^2 2x \cos^2 2x$.

(*vi*) $\dfrac{1-\tan^2\theta}{1+\tan^2\theta} = \cos 2\theta$.

(*vii*) $\tan\dfrac{t}{2} = \dfrac{1-\cos t}{\sin t} = \dfrac{\sin t}{1+\cos t}$

(*viii*) If $u = \tan\dfrac{x}{2}$ then $\sin x = \dfrac{2u}{1+u^2}$, $\cos x = \dfrac{1-u^2}{1+u^2}$.

(*ix*) $\dfrac{2}{1+\cos 2t} = \sec^2 t$.

(*x*) $\cos^4\theta - \sin^4\theta = \cos 2\theta$.

(*xi*) $\dfrac{1+\sin 2A + \cos 2A}{1+\sin 2A - \cos 2A} = \cot A$.

35. Prove that:

$\cos (A+B) \cos (A-B) = \cos^2 A - \sin^2 B$

36. $\dfrac{\tan (A+B)}{\cot (A-B)} = \dfrac{\sin^2 A - \sin^2 B}{\cos^2 A - \sin^2 B}$

37. $\dfrac{1+\sin 2A - \cos 2A}{1+\sin 2A + \cos 2A} = \tan A$

Hint : $\text{LHS} = \dfrac{1+2\sin A\cos A - (1-2\sin^2 A)}{1+2\sin A\cos A + (2\cos^2 A - 1)}$

38. If $\tan\theta = \frac{b}{a}$ then $a\cos^2\theta + b/2\sin 2\theta = a$.

$$a\cos\theta + b\sin\theta = \sqrt{a^2+b^2}\left(\frac{a\cos\theta}{\sqrt{a^2+b^2}} + \frac{b\sin\theta}{\sqrt{a^2+b^2}}\right)$$

$$= \sqrt{a^2+b^2}\,(\sin\alpha\cos\theta + \cos\alpha\sin\theta)$$

$$= \sqrt{a^2+b^2}\,\sin(\theta+\alpha),\text{ where }\tan\alpha = \frac{b}{a}.$$

39. $\frac{\sec 8A - 1}{\sec 4A - 1} = \frac{\tan 8A}{\tan 2A}$ (**Note:** $\sin 8A = 2\sin 4A\cos 4A$)

(**Hint:** LHS $= \frac{1-\cos 8A}{\cos 8A}\frac{\cos 4A}{1-\cos 4A} = \frac{2\sin^2 4A\cos 4A}{\cos 8A\cdot 2\sin^2 2A}$

$= \frac{\sin 4A\cdot 2\sin 4A\cdot\cos 4A}{\cos 8A\cdot 2\sin^2 2A}$.)

40. If $A + B + C = \pi/2$ show

$$\sin^2 A + \sin^2 B + \sin^2 C = 1 - 2\sin A\sin B\sin C$$

41. If $A + B = C$ then show

$$\cos^2 A + \cos^2 B + \cos^2 C = 1 + 2\cos A\cos B\cos C$$

42. Show $\sin^2 30^\circ \sin 45^\circ \cos 45^\circ = \frac{1}{8}$.

43. $\frac{\cos 6A + 6\cos 4A + 15\cos 2A + 10}{\cos 5A + 5\cos 3A + 10\cos A} = 2\cos A$.

[**Hint:** $2\cos A(\cos 5A + 5\cos 3A + 10\cos A)$]

$= (\cos 6A + \cos 4A) + 5(\cos 4A + \cos 2A) + 10(\cos 2A + 1)$

$= \cos 6A + 6\cos 4A + 15\cos 2A + 10$

NOTES

1.13 FURTHER READING

Khanna, V.K, S.K. Bhambri, C.B. Gupta and Vijay Gupta. *Quantitative Techniques.* New Delhi: Vikas Publishing House.

Khanna, V.K, S.K. Bhambri and Quazi Zameeruddin. *Business Mathematics.* New Delhi: Vikas Publishing House.

UNIT 2 DIFFERENTIATION

NOTES

Structure

2.0 INTRODUCTION

In Mathematics, differentiation refers to the act of finding derivatives. Derivative is defined as the instantaneous rate of change of a function. The derivative gives the slope of the tangent to the graph of the function at a point. Thus, it is a mathematical formulation of the rate of change.

Differentiation expresses the rate of change of any quantity y with respect to the change in another quantity x, with which it has a functional relationship.

2.1 UNIT OBJECTIVES

After going through this unit, you will be able to:

- Understand differentiation
- Understand how to check the continuity of functions
- Learn about differential coefficient
- Understand algebra of differentiable functions
- Analyse tangent and normal derivatives

- Explain the differentiation of implicit functions and parametric forms
- Define partial differentiation
- Learn about maxima and minima of functions

NOTES

2.2 LIMITS AND CONTINUITY (WITHOUT PROOF)

2.2.1 Functions and their Limits

In Mathematics, you usually deal with two kinds of quantities, namely constants and variables. A quantity which is liable to vary is called a *variable quantity* or simply a *variable*. Temperature, pressure, distance of a moving train from a station are all variable quantities. On the other hand, a quantity that retains its value through all mathematical operations is termed as a *constant quantity* or a *constant*. Numbers like 4, 5, 2.5, π, etc., are all constants.

If x is a real variable (i.e., x takes up different values that are real numbers). Then in quantities $\log x$, $\sin x$, x^2, etc., log, sin, square are the functions.

Let us write $y = x^2$.

Therefore, if $x = 2$, then $y = 4$, if $x = 3$, then $y = 9$, etc.

Thus, for each value of x, y gets a unique corresponding value and this value is assigned each time by a certain *rule* (namely, square). This rule is what we call a function.

So, in general, by a function of x, we mean a rule that gives us a unique value corresponding to each value of x.

So, if $y = \sin x$, then whenever we give a different value to x, we get a corresponding unique value for y with the assistance of the function sine.

When we are dealing with any function, we simply write

$$y = f(x)$$

and say that y is a function of x although to be very correct we should say that y is the value assigned by the function f corresponding to a value of x. And we call x as the independent variable and y as dependent variable.

Functions play important role in Mathematics, Physics and Social Sciences.

A function which assigns a *fixed* value for every value of x is called a *constant function*. For example, $f(x) = 3$ is a constant function, since for any value of x, $f(x)$ remains equal to 3.

The next important notion is that of the limit of a function. It is quite possible that $f(x)$ may not be defined for all values of x. As an illustration, consider $f(x) = \dfrac{x^2 - 25}{x - 5}$. This is a function of x, provided x takes all real values except 5. If it were defined at $x = 5$, we would have got $f(5) = \frac{25-25}{5-5} = \frac{0}{0}$; a meaningless quantity. $\frac{0}{0}$ can take any finite value whatsoever, like $\frac{0}{0} = 5$. $\frac{0}{0} = \pi$, etc., since $0 \times 5 = 0$, $0 \times \pi = 0$. One could perhaps say here that why don't we cancel $x - 5$ first and then put $x = 5$ to get $f(5)$ equal to 10. There is a lapse in this argument as $x - 5$ is zero when $x = 5$ and cancellation of zero factor is not allowed in Mathematics, because you would get very absurd results like $1 = 2$ and $3 = 15$, etc., since $0 \times 1 = 0 \times 2$ and $0 \times 3 = 0 \times 15$. As a consequence you cannot determine $f(5)$, the value of $f(x)$ at $x = 5$. But you should not leave the problem here. Instead you try to evaluate the value of $f(x)$ when x is *very* near to 5 (and this will finally lead you to a value that would *almost* be the value of $f(5)$).

NOTES

Thus, you can evaluate $f(x)$ at $x = 4.9998$ or $x = 5.00001$. The technique is quite simple. Cancel x–5 first (this step is perfectly legitimate as x is not equal to 5); then substitute the value of x. For example, $f(4.9998) = 4.9998 + 5\ 9.9998$ and $f(5.00001) = 5.00001 + 5 = 10.00001$.

We now write down some of the values given to x and the corresponding values acquired by $f(x)$ in Tables 2.1 and 2.2. In first table values of x are *increasing upto* 5 (being always less than 5) and in second table values of x are *decreasing down* to 5 (being always greater than 5).

Table 2.1 *Increasing Value of x*

Value of x	*Value of f (x)*
4	9
4.235	9.235
4.976	9.976
4.99998	9.99998
4.9999999	9.999999
↓	↓

Table 2.2 *Decreasing Value of x*

Value of x	*Value of f (x)*
7	12
6.31	11.31
5.7984	10.7984
5.2175	10.2175
5.0039	10.0039
5.0000001	10.0000001
↓	↓

The above is expressed mathematically as: x tends to 5 from the left (in Table 2.1) and x tends to 5 from the right (in Table 2.2). In the first case we write $x \to 5-$ and in the second case we write $x \to 5+$.

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Observe the pattern of change in the second column of each table. One could, after slight concentration see that in the first situation $f(x) \to 10-$ while in the second case $f(x) \to 10+$. Thus, you are tempted to assert that $f(x)$ approaches 10 (both from left and from right) as x approaches 5. This number 10, we call limit of $f(x)$ as x approaches 5.

This fact is denoted by, $\lim\limits_{x \to 5} f(x) = 10$.

Consider now any small positive number, say, 0.01. When $| x - 5 | < 0.01$, i.e., $-0.01 < x - 5 < 0.01$ or $4.99 < x < 5.01$, then $-0.01 < f(x) - 10 < 0.01$

Or, $\quad | f(x) - 10 | < 0.01$

($4.99 < x < 5.01 \quad$ and $\quad x \neq 5 \quad \Rightarrow \quad f(x) = x + 5$

This yields in turn $f(x) - 10 = x - 5$

So, $\quad -0.01 < f(x) - 10 < 0.01$)

One can repeat the above experiment by starting with another small positive number say 0.00002 and note that whenever $| x - 5 | < 0.00002$,

Then, $\quad | f(x) - 10 | < 0.00002$.

The expected conclusion will be that, however a small positive number ε we may start with, we shall always be able to find a $\delta > 0$ such that whenever $| x - 5 | < \delta$ then $| f(x) - 10 | < \varepsilon$. In the above illustration $\delta = \varepsilon$ will suffice.

Thus, you are led to the following definition of limit.

You say that $\lim\limits_{x \to a} f(x) = l$ if corresponding to any $\varepsilon > 0$, we can find $\delta > 0$, such that $| f(x) - l | < \varepsilon$ whenever $| x - a | < \delta$.

Let us evaluate some limits using this definition. Later on we'll give other convenient methods too.

Example 2.1: Evaluate $\lim\limits_{x \to a} x$.

Solution: Let $\varepsilon > 0$ be any number.

Take, $\quad \delta = \varepsilon$. Evidently $\delta > 0$.

Now, $\quad | x - a | < \delta \Rightarrow | f(x) - a | < \varepsilon$

Since, $\quad f(x) = x$.

Hence, $\lim\limits_{x \to a} a$.

Example 2.2: Evaluate $\lim_{x \to a} x^2$.

Solution: Let $\varepsilon > 0$ be any number.

Take, $\delta = -2 + \sqrt{4 + \varepsilon}$. Clearly $\delta > 0$

Now, $|x + 2| = |x - 2 + 4| \leq |x - 2| + 4 < \delta + 4$

So, $|x - 2| < \delta \Rightarrow |x^2 - 4| = |x + 2|\ |x - 2| < d^2 + 4\delta = \varepsilon$

Hence, $\lim_{x \to 2} x^2 = 4$.

NOTES

Example 2.3: Determine $\lim_{x \to 3} \frac{1}{x}$.

Solution: Let $\varepsilon > 0$.

Put, $\delta = \frac{9\varepsilon}{1 + 3\varepsilon}$

Now, $|x - 3| < \delta \Rightarrow |x - 3| < \frac{9\varepsilon}{1 + 3\varepsilon}$ ...(1)

Also, $|x - 3| < \delta \Rightarrow -\delta < x - 3$

$\Rightarrow$ $x > 3 - \delta = 3 - \frac{9\varepsilon}{1 + 3\varepsilon} = \frac{3}{1 + 3\varepsilon}$

$\Rightarrow$ $\frac{1}{x} < \frac{1 + 3\varepsilon}{3}$ ...(2)

Equations (1) and (2) imply

$$\left|\frac{1}{x} - \frac{1}{3}\right| = \left|\frac{x - 3}{3x}\right| < \frac{9\varepsilon}{1 + 3\varepsilon}\ \frac{1 + 3\varepsilon}{3} \cdot \frac{1}{3} = \varepsilon$$

Consequently, $\lim_{x \to 3} \frac{1}{x} = \frac{1}{3}$.

Example 2.4: Find out the limit of $\frac{x^2 - 1}{x - 1}$ *as* $x \to 1$.

Solution: Let $\varepsilon > 0$ be any number.

Take $\delta = \varepsilon$.

Now $|x - 1| < \delta \Rightarrow |x - 1| < \varepsilon$

$\Rightarrow |x + 1 - 2| < \varepsilon$

$\Rightarrow \left|\frac{x^2 - 1}{x - 1} - 2\right| < \varepsilon$

$\Rightarrow \lim_{x \to 1} \frac{x^2 - 1}{x - 1} = 2.$

2.2.2 *h*-Method for Determining Limits

You put $a + h$ in place of x and simplify such that h gets cancelled from denominator and numerator. Putting $h = 0$, you get limit of $f(x)$ as $x \to a$.

NOTES

Example 2.5: Evalute $\lim\limits_{x\to 1} \dfrac{x^2 - 1}{x^2 - 1}$.

Solution:
$$\lim_{x\to 1} \frac{x^3 - 1}{x^2 - 1} = \lim_{h\to 0} \frac{(1+h)^3 - 1}{(1+h)^2 - 1}$$

$$= \lim_{h\to 0} \frac{3h + 3h^2 + h^3}{2h + h^2}$$

$$= \lim_{h\to 0} \frac{3 + 3h + h^2}{2 + h}$$

$$= \frac{3}{2}.$$

2.2.3 Expansion Method for Evaluating Limits

This method is applicable to the functions which can be expanded in series. Following expansions are often utilized.

(1) $(1 + x)^n = 1 + nx + \dfrac{n(n-1)x^2}{2!} + \dfrac{n(n-1)(n-1)x^3}{3!} + \ldots$ provided $|x| < 1$

and n is any real number.

***Note*:** If n is a positive integer, the expansion on RHS has finite number of terms only.

(2) $\log(1 + x) = x - \dfrac{x^2}{2} + \dfrac{x^3}{3} - \dfrac{x^4}{4} + \ldots \infty$ provided $|x| < 1$.

(3) $e^x = 1 + x + \dfrac{x^2}{2!} + \dfrac{x^3}{3!} + \ldots \infty$

(4) $\sin x = x - \dfrac{x^3}{3!} + \dfrac{x^5}{5!} \ldots \infty$

(5) $\cos x = 1 - \dfrac{x^2}{2!} + \dfrac{x^4}{4!} \ldots \infty$

The method will be illustrated by means of the following examples.

Example 2.6: *Show that* $\lim\limits_{x\to 0} \dfrac{\sin x}{x} = 1$.

Solution:
$$\lim_{x\to 0} \frac{\sin x}{x} = \lim_{x\to 0} \frac{x - \dfrac{x^3}{3!} + \dfrac{x^5}{5!} \ldots}{x}$$

$$= \lim_{x\to 0}\left(1-\frac{x^2}{3!}+\frac{x^4}{5!}\cdots\right) = 1.$$

***Note*:** The result of Example 2.6 shall be frequently used later on.

NOTES

Example 2.7: Evaluate $\lim\limits_{x\to 0}\dfrac{e^x - e^{-x}}{x}$.

Solution: $\lim\limits_{x\to 0}\dfrac{e^x - e^{-x}}{x}$

$$= \lim_{x\to 0}\frac{\left(1+x+\frac{x^2}{2!}+\frac{x^3}{3!}\cdots\right)-\left(1-x+\frac{x^2}{2!}-\frac{x^3}{3!}\cdots\right)}{x}$$

$$= \lim_{x\to 0}\frac{2\left(x+\frac{x^3}{3!}+\frac{x^5}{5!}\cdots\right)}{x}$$

$$= \lim_{x\to 0} 2\left(1+\frac{x^2}{3!}+\frac{x^4}{5!}+\ldots\right)$$

$$= 2.$$

***Notes*:**

1. $\lim\limits_{x\to 0} f(x) = l$ if and only if

   $\lim\limits_{x\to a-} f(x) = l = \lim\limits_{x\to a+} f(x)$.

   If one of the two equalities fails to hold, then we say that limit of $f(x)$ as $x \to a$ does not exist.

   Consider $\lim\limits_{x\to 0}\frac{1}{x}$. It can be easily seen that, if $x \to 0-$, then $\frac{1}{x} \to -\infty$ while, if $x \to 0+$, then $\frac{1}{x} \to \infty$. So, $\lim\limits_{x\to 0}\frac{1}{x}$ does not exist.

2. If $\lim\limits_{x\to a} f(x)$ exists, then it must be unique.

Example 2.8: *f(x)* is defined as under

$$f(x) = 0 \qquad \text{for } x \leq 0$$

$$= \frac{1}{2} - x \qquad \text{for } x > 0.$$

Show that $\lim\limits_{x\to 2} f(x)$ does not exist.

Solution: $\lim\limits_{x\to 0-} f(x) = \lim\limits_{h\to 0} f(0-h) = 0.$

Also, $\lim\limits_{x\to 0+} f(x) = \lim\limits_{h\to 0+} f(0+h) = \lim\limits_{h\to 0} f\left(\frac{1}{2}-h\right)$

$$= \frac{1}{2}$$

Since, $\lim\limits_{x\to 0-} f(x) \neq \lim\limits_{x\to 0+} f(x)$,

$\lim\limits_{x\to 0} f(x)$ does not exist.

NOTES

Notes**:** 1. If $\lim\limits_{x\to a} f(x) = l$ and $\lim\limits_{x\to a} g(x) = m$, then

(*i*) $\lim\limits_{x\to a} [f(x) \pm g(x)] = l \pm m$

(*ii*) $\lim\limits_{x\to a} [f(x)\, g(x)] = lm$

(*iii*) $\lim\limits_{x\to a} \dfrac{f(x)}{g(x)} = \dfrac{l}{m}$ provided $m \neq 0$

(*iv*) $\lim\limits_{x\to a} [f(x)]^{g(x)} = l^m$, provided l^m is defined.

2. If $f(x) < g(x)$ for all x, then $\lim\limits_{x\to a} f(x) \leq \lim\limits_{x\to a} g(x)$.

2.2.4 Continuous Functions

We have seen in Example 2.2 that $\lim\limits_{x\to 2} x^2 = 4$, which is same as value of x^2 at $x = 2$.

Whereas, in Example 2.4, $\lim\limits_{x\to 1} \dfrac{x^2-1}{x-1} = 2$, but the function itself is not defined as $x = 1$.

Again consider,
$$f(x) = \frac{x^2-9}{x-3}, \; x \neq 3$$
$$= 1, \qquad x = 3$$

In this case, $\lim\limits_{x\to 3} \dfrac{x^2-9}{x-3} = 6$, and $f(3) = 1$.

Thus, a function may possess a limit as $x \to a$ but it may or may not be defined at $x = a$. Even if it is defined at $x = a$, its value may not be equal to its limit. This prompts us to define the following type of functions.

A function $f(x)$ is said to be continuous at $x = a$, if $\lim\limits_{x\to a} f(x) = f(a)$.

In other words, $f(x)$ is said to be continuous at $x = a$, if given $\varepsilon > 0$, there exists $\delta > 0$, such that $|f(x) - f(a)| < \varepsilon$, whenever $|x - a| < \delta$.

Example 2.9: Check for continuity at $x = 0$, the function $(x) = |x|$.

Solution: By definition of absolute value, we can write.

$$f(x) = x \text{ for all } x \geq 0$$
$$= -x \text{ for all } x < 0$$

We note that, $f(0) = 0$

Further, $\lim\limits_{x\to 0-} f(x) = \lim\limits_{h\to 0} f(0-h) = \lim\limits_{h\to 0} -(-h) = 0$

Also, $\lim\limits_{x\to 0+} f(x) \; \lim\limits_{h\to 0} f(0+h) = \lim\limits_{h\to 0} h = 0.$

Thus, $\lim\limits_{h\to 0} f(x) = 0 = f(0)$

Hence, $f(x)$ is continuous at $x = 0$.

NOTES

2.3 DIFFERENTIATION AND DIFFERENTIAL COEFFICIENT

Let y be a function of x. We call x an independent variable and y dependent variable.

Note: There is no sanctity about x being independent and y being dependent. This depends upon which variable we allow to take any value, and then corresponding to that value, determine the value, of the other variable. Thus in $y = x^2$, x is an independent variable and y a dependent, whereas the same function can be re-written as $x = \sqrt{y}$. Now y is an independent variable, and x is a dependent variable. Such an 'inversion' is not always possible.

For example, in $y = \sin x + x^3 + \log x + x^{1/2}$, it is rather impossible to find x in terms of y.

Differential coefficient of $f(x)$ with respect to x

Let, $$y = f(x) \qquad \text{...(2.1)}$$

and let x be changed to $x + \delta x$. If the corresponding change in y is δy, then

$$y + \delta y = f(x + \delta x) \qquad \text{...(2.2)}$$

Equations (2.1) and (2.2) imply that

$$\delta y = f(x + \delta x) - f(x)$$

$$\Rightarrow \qquad \frac{\delta y}{\delta x} = \frac{f(x+\delta x) - f(x)}{\delta x}$$

$\lim\limits_{\delta x\to 0} \dfrac{f(x+\delta x) - f(x)}{\delta x}$, if it exists, is called the *differential coefficient* of y with respect to x and is written as $\dfrac{dy}{dx}$.

Thus, $$\frac{dy}{dx} = \lim_{\delta x\to 0} \frac{\delta y}{\delta x} = \lim_{\delta x\to 0} \frac{f(x+\delta x) - f(x)}{\delta x}.$$

Let $f(x)$ be defined at $x = a$. The derivative of $f(x)$ at $x = a$ is defined as $\lim\limits_{h\to 0} \dfrac{f(a+h) - f(a)}{h}$, provided the limit exists, and then it is written as $f'(a)$ or $\left(\dfrac{dy}{dx}\right)_{x=a}$. We sometimes write the definition in the form $f'(a) = \lim\limits_{x\to a} \dfrac{f(x) - f(a)}{x - a}$.

Note: $f'(a)$ can also be evaluated by first finding out $\frac{dy}{dx}$ and then putting in it, $x = a$.

NOTES

Notation: $\frac{dy}{dx}$ is also denoted by y' or y_1 or Dy or $f'(x)$ in case $y = f(x)$.

Example 2.10: Find $\frac{dy}{dx}$ and $\left(\frac{dy}{dx}\right)_{x=3}$ for $y = x^3$.

Solution: We have $y = x^3$

Let δx be the change in x and let the corresponding change in y be δy.

Then, $$y + \delta y = (x + \delta x)^3$$

$$\Rightarrow \quad \delta y = (x + \delta x)^3 - y = (x + \delta x)^3 - x^3$$
$$= 3x^2 (\delta x) + 3x (\delta x)^2 + (\delta x)^3$$

$$\Rightarrow \quad \frac{\delta y}{\delta x} = 3x^2 + 3x (\delta x) + (\delta x)^2$$

Consequently, $$\frac{dy}{dx} = \lim_{\delta x \to 0} \frac{\delta y}{\delta x} = 3x^2$$

Also, $$\left(\frac{dy}{dx}\right)_{x=3} = 3 \,.\, 3^2 = 27$$

Example 2.11: Show that for $y = |x|$, $\frac{dy}{dx}$ does not exist at $x = 0$.

Solution: If $\frac{dy}{dx}$ exists at $x = 0$, then

$$\lim_{h \to 0} \frac{f(0+h) - f(0)}{h} \text{ exists.}$$

So, $$\lim_{h \to 0+} \frac{f(0+h) - f(0)}{h} = \lim_{h \to 0-} \frac{f(0-h) - f(0)}{-h}$$

Now, $$f(0+h) = |h|,$$

So, $$\lim_{h \to 0+} \frac{f(0+h) - f(0)}{h} = \lim_{h \to 0+} \frac{|h| - 0}{h} = \lim_{h \to 0+} \frac{h}{h} = 1$$

Also, $$\lim_{h \to 0-} \frac{f(0-h) - f(0)}{-h} = \lim_{h \to 0-} \frac{|-h| - 0}{-h}$$

$$= \lim_{h \to 0-} \frac{h}{-h} = -1$$

Hence, $$\lim_{h \to 0+} \frac{f(0+h) - f(0)}{h} \neq \lim_{h \to 0-} \frac{f(0-h) - f(0)}{-h}$$

Consequently, $\lim_{h \to 0} \frac{f(0+h) - f(0)}{h}$ does not exist.

Notes: 1. A function $f(x)$ is said to be derivable or differentiable at $x = a$ if its derivative exists at $x = a$.

2. A differentiable function is necessarily continuous.

Proof: Let $f(x)$ be differentiable at $x = a$.

Then $\lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$ exists, say, equal to l.

$$\lim_{h \to 0} \frac{f(a+h) - f(a)}{h} = 0$$

$$\Rightarrow \lim_{h \to 0} [f(a+h) - f(a)] = 0$$

$$\Rightarrow \lim_{h \to 0} f(a+h) = f(a)$$

$$\Rightarrow \lim_{h \to 0} f(x) = f(a)$$

h can be positive or negative.

In other words $f(x)$ is continuous at $x = a$.

3. Converse of the statement in Note 2 is not true in general.

NOTES

Example 2.12: Show that $f(x) = x^2 \sin \frac{1}{x}$, $x \neq 0$ and $x = 0$ is differentiable at $x = 0$.

Solution: $\lim_{h \to 0} \frac{f(h) - f(0)}{h}$

$$= \lim_{h \to 0} \frac{\left(h^2 \sin \frac{1}{h} - 0\right)}{h}$$

$$= \lim_{h \to 0} \left(h \sin \frac{1}{h}\right)$$

Now, $\left|\sin \frac{1}{h}\right| < 1$ and so $\left|h \sin \frac{1}{h}\right| < |h|$

$$\Rightarrow \qquad \lim_{h \to 0} \left|h \sin \frac{1}{h}\right| \leq \lim_{h \to 0} |h| = 0$$

Hence, $\qquad \lim_{h \to 0} \frac{f(h) - f(0)}{h} = 0$

i.e., $\qquad f(x)$ is differentiable at $x = 0$

And $\qquad f'(0) = 0.$

2.4 DERIVATIVES OF FUNCTIONS

NOTES

2.4.1 Algebra of Differentiable Functions

We will now prove the following results for two differentiable functions $f(x)$ and $g(x)$.

(1) $\frac{d}{dx}[f(x) \pm g(x)] = f'(x) \pm g'(x)$

(2) $\frac{d}{dx}[f(x) \cdot g(x)] = f'(x)\, g(x) + f(x)\, g'(x)$

(3) $\frac{d}{dx}\left[\frac{f(x)}{g(x)}\right] = \frac{f'(x)g(x) - f(x)\,g'(x)}{[g(x)]^2}$

(4) $\frac{d}{dx}[cf(x)] = cf'(x)$, where c is a constant

Where, of course, by $f'(x)$ mean $\frac{d}{dx}f(x)$.

Proof:

(1) $\frac{d}{dx}[f(x) + g(x)]$

$$= \lim_{\delta x \to 0} \frac{[f(x+\delta x) + g(x+\delta x)] - [f(x) + g(x)]}{\delta x}$$

$$= \lim_{\delta x \to 0} \left[\frac{f(x+\delta x) - f(x)}{\delta x} + \frac{g(x+\delta x) - g(x)}{\delta x}\right]$$

$$= \lim_{\delta x \to 0} \frac{f(x+\delta x) - f(x)}{\delta x} + \lim_{\delta x \to 0} \frac{g(x+\delta x) - g(x)}{\delta x}$$

$$= f'(x) + g'(x)$$

Similarly, it can be shown that

$$\frac{d}{dx}[f(x) - g(x)] = f'(x) - g'(x)$$

Thus, we have the following rule:

The derivative of the sum (or difference) of two functions is equal to the sum (or difference) of their derivatives.

NOTES

(2) $\dfrac{d}{dx}[f(x)\ g(x)]$

$$= \lim_{\delta x \to 0} \frac{f(x+\delta x)\ g(x+\delta x) - f(x)\ g(x)}{\delta x}$$

$$= \lim_{\delta x \to 0} \frac{g(x+\delta x)\big[f(x+\delta x) - f(x)\big] + f(x)\big[g(x+\delta x) - g(x)\big]}{\delta x}$$

$$= \lim_{\delta x \to 0} \left[g(x+\delta x).\left[\frac{f(x+\delta x) - f(x)}{\delta x}\right] + f(x)\left[\frac{g(x+\delta x) - g(x)}{\delta x}\right]\right]$$

$$= [\lim_{\delta x \to 0} g\ (x+\delta x)] \lim_{\delta x \to 0} \left[\frac{f(x+\delta x) - f(x)}{\delta x}\right]$$

$$+ \left[\lim_{\delta x \to 0} f\ (x)\right]\left[\lim_{\delta x \to 0} \frac{g(x+\delta x) - g(x)}{\delta x}\right]$$

$$= g\ (x)\, f'\ (x) + f\,(x)\ g'(x).$$

Thus, we have the following rule for the derivative of a product of two functions:

The derivative of a product of two functions = (*the derivative of first function* × *second function*) + (*first function* × *derivative of second function*).

(3) $\dfrac{d}{dx}\left[\dfrac{f(x)}{g(x)}\right]$

$$= \lim_{\delta x \to 0} \frac{\dfrac{f(x+\delta x)}{g(x+\delta x)} - \dfrac{f(x)}{g(x)}}{\delta x}$$

$$= \lim_{\delta x \to 0} \frac{f(x+\delta x)\ g(x) - f(x)\ g(x+\delta x)}{\delta x . g\ (x+\delta x)\ g(x)}$$

$$= \lim_{\delta x \to 0} \frac{g(x)[f(x+\delta x) - f(x)] - f(x)[g(x+\delta x) - g(x)]}{\delta x . g(x+\delta x) g(x)}$$

$$= [\lim_{\delta_x \to 0} \frac{1}{g(x+\delta x)} \cdot \frac{1}{g(x)}] \left[\lim_{\delta x \to 0} \frac{g(x)[f(x+\delta x) - f(x)]}{\delta x}\right.$$

$$\left. - \lim_{\delta x \to 0} \frac{f(x)[g(x+\delta x) - g(x)]}{\delta x}\right]$$

$$= \frac{1}{[g(x)]^2} \cdot [g(x) f'(x) - f(x) g'(x)]$$

$$= \frac{f'(x) g(x) - f(x) g'(x)}{[g(x)]^2}.$$

The corresponding rule is stated as under:

The derivative of quotient of two functions =

$$\frac{(\textit{Derivative of Numerator} \times \textit{Denominator}) - (\textit{Numerator} \times \textit{Derivative of Denominator})}{(\textit{Denominator})^2}$$

$$(4)\ \frac{d}{dx}[cf(x)] = \lim_{\delta x \to 0} \frac{cf(x+\delta x) - cf(x)}{\delta x}$$

$$= c \lim_{\delta x \to 0}\left[\frac{f(x+\delta x) - f(x)}{\delta x}\right] = cf'(x).$$

NOTES

The derivative of a constant function is equal to the constant multiplied by the derivative of the function.

2.4.2 Differential Coefficients of Standard Functions

I. $\boldsymbol{\frac{d}{dx}(x^n) = nx^{n-1}}$

Proof: Let, $y = x^n$

Then, $(y + \delta y) = (x + \delta x)^n$

$\Rightarrow$ $\delta y = (x + \delta x)^n - y = (x + \delta x)^n - x^n$

$$= x^n\left[\left(1+\frac{\delta x}{x}\right)^n - 1\right]$$

$$= x^n\left[1 + n\left(\frac{\delta x}{x}\right) + \frac{n(n-1)}{2!}\left(\frac{\delta x}{x}\right)^2 + \ldots - 1\right]$$

$$= nx^{n-1}(\delta x) + \frac{n(n-1)}{2!}x^{n-2}(\delta x)^2 + \ldots$$

$$\frac{\delta y}{\delta x} = nx^{n-1} + \text{terms containing powers of } \delta x$$

$\Rightarrow$ $$\lim_{\delta x \to 0}\frac{\delta y}{\delta x} = nx^{n-1}$$

Hence, $$\frac{dy}{dx} = nx^{n-1}.$$

II. (*i*) $\boldsymbol{\frac{d}{dx}(a^x) = a^x \log_e a}$

(*ii*) $\boldsymbol{\frac{d}{dx}(e^x) = e^x}$

Proof: (*i*) Let, $y = a^x$

then, $y + \delta y = a^{x+\delta x}$

$\Rightarrow$ $\delta y = a^{x+\delta x} - a^x = a^x(a^{\delta x} - 1)$

$\Rightarrow$ $$\frac{\delta y}{\delta x} = \frac{a^x(a^{\delta x} - 1)}{\delta x}$$

$\Rightarrow$ $$\frac{dy}{dx} = \lim_{\delta x \to 0}\frac{\delta y}{\delta x} = a^x \lim_{\delta x \to 0}\left(\frac{a^{\delta x} - 1}{\delta x}\right)$$

$$= a^x \lim_{\delta x \to 0}\frac{\left[1 + \delta x(\log a) + \frac{(\delta x)^2(\log a)^2}{2} + \ldots - 1\right]}{\delta x}$$

$$= a^x \lim_{\delta x \to 0}(\log a + \text{terms containing } \delta x)$$

$$= a^x \log a = a^x \log_e a$$

proves the first part.

(ii) Since $\log_e e = 1$, it follows from result (*i*) that $\frac{d}{dx} e^x = e^x$.

III. $\frac{d}{dx}\log_e x = \frac{1}{x}$

NOTES

Proof: Let, $y = \log x$

$\Rightarrow \quad y + \delta y = \log (x + \delta x)$

$\Rightarrow \quad \delta y = \log (x + \delta x) - \log x = \log\left(\frac{x+\delta x}{x}\right)$

$$\Rightarrow \quad \frac{\delta y}{\delta x} = \frac{\log\left(1+\frac{\delta x}{x}\right)}{\delta x}$$

$$= \frac{1}{x}\cdot\frac{x}{\delta x}\log\left(1+\frac{\delta x}{x}\right) = \frac{1}{x}\log\left(1+\frac{\delta x}{x}\right)^{x/\delta x}$$

$$\Rightarrow \quad \lim_{\delta x \to 0}\frac{\delta y}{\delta x} = \frac{1}{x}\lim_{\delta x\to 0}\log_e\left(1+\frac{\delta x}{x}\right)^{x/\delta x}$$

$$= \frac{1}{x}\log_e e = \frac{1}{x}$$

as $\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n = e$ and $\log_e e = 1$

Hence, $\frac{dy}{dx} = \frac{1}{x}$

IV. $\frac{d}{dx}(\sin x) = \cos x$

Proof: Now, $y = \sin x \Rightarrow y + \delta y = \sin (x + \delta x)$

$$\Rightarrow \quad \delta y = \sin (x + \delta x) - \sin x = 2\cos\left(x+\frac{\delta x}{2}\right)\sin\frac{\delta x}{2}$$

$$\Rightarrow \quad \frac{\delta y}{\delta x} = \frac{2\cos\left[x+\frac{\delta x}{2}\right]\sin\frac{\delta x}{2}}{\delta x} = \cos\left(x+\frac{\delta x}{2}\right)\cdot\left(\frac{\sin\frac{\delta x}{2}}{\frac{\delta x}{2}}\right)$$

$$\Rightarrow \quad \lim_{\delta x\to 0}\frac{\delta y}{\delta x} = \lim_{\delta x\to 0}\left[\cos\left(x+\frac{\delta x}{2}\right)\right]\left[\lim_{\delta x\to 0}\frac{\sin\frac{\delta x}{2}}{\frac{\delta x}{2}}\right]$$

$$= (\cos x)\left[\lim_{\delta x\to 0}\frac{\sin\frac{\delta x}{2}}{\frac{\delta x}{2}}\right] = (\cos x)(1) = \cos x$$

$$\Rightarrow \quad \frac{dy}{dx} = \cos x.$$

V. $\frac{d}{dx}(\cos x) = -\sin x$ [The proof is similar to that of (IV).]

Notes:

NOTES

1. The technique employed in the proofs of (I) to (IV) above is known as '*ab initio*' technique. We have utilized (apart from simple formulas of Algebra and Trigonometry) the definition of differential coefficient only. We have nowhere used the algebra of differentiable functions.
2. In (VI) to (XII) we shall utilize the algebra of differentiable functions.

VI. $\frac{d}{dx}(c) = 0$**, where** c **is a constant.**

Proof: Let, $y = c = cx^0.$

Then, $$\frac{dy}{dx} = c\left(\frac{dx^0}{dx}\right) = c(0.x^{0-1}) = 0.$$

VII. $\frac{d}{dx}(\tan x) = \sec^2 x$

Proof: Let, $y = \tan x = \frac{\sin x}{\cos x}$

$$\frac{dy}{dx} = \frac{\frac{d}{dx}(\sin x)\cos x - \sin x\frac{d}{dx}(\cos x)}{(\cos x)^2}$$

$$= \frac{(\cos x)(\cos x) - \sin x(-\sin x)}{(\cos x)^2}$$

$$\text{s} = \frac{\cos^2 x + \sin^2 x}{\cos^2 x} = \frac{1}{\cos^2 x} = \sec^2 x.$$

VIII. $\frac{d}{dx}(\sec x) = \sec x \tan x$

Proof: Let, $y = \sec x = \frac{1}{\cos x}$

Then, $$\frac{dy}{dx} = \frac{\frac{d}{dx}(1)\cos x - (1)\frac{d}{dx}(\cos x)}{(\cos x)^2}$$

$$= \frac{(0)(\cos x) - (-\sin x)}{\cos^2 x}$$

$$= \frac{\sin x}{\cos^2 x} = \frac{\sin x}{\cos x}\cdot\frac{1}{\cos x} = \tan x \sec x.$$

Before we proceed further, we will introduce hyperbolic functions.

We define hyperbolic sine of x as $\frac{e^x - e^{-x}}{2}$ and write it as

$\sin h\ x = \frac{e^x - e^{-z}}{2}.$

Hyperbolic cosine of x is defined to be $\frac{e^x + e^{-x}}{2}$ and is denoted by $\cos h\, x$.

It can be easily verified that

$$\cos h^2 x - \sin h^2 x = 1$$

Since $(\cos h\,\theta, \sin h\,\theta)$ satisfies the equation $x^2 - y^2 = 1$ of a hyperbola, these functions are called hyperbolic functions.

In analogy with circular functions (i.e., $\sin x$, $\cos x$, etc.) we define $\tan h\, x$, $\cot h\, x$ $\sec h\, x$ and $\operatorname{cosec} h\, x$.

Thus, by definition, $\tan h\ x = \frac{\sin h\, x}{\cos h\, x}$, $\cot h\, x = \frac{1}{\tan h\, x}$,

$$\sec h\, x = \frac{1}{\cos h\, x} \text{ and } \operatorname{cosec} h\, x = \frac{1}{\sin h\, x}.$$

IX. $\boldsymbol{\frac{d}{dx}(\sin h\, x) = \cos h\, x}$

Proof: Before proving this result, we say that

$$\frac{d}{dx}(e^{-x}) = -e^{-x},$$

Because,

$$e^{-x} = (e^{-1})^x$$

$$\Rightarrow \qquad \frac{d(e^{-x})}{dx} = (e^{-1})^x \log_e (e^{-1}) = e^{-x}(-1) = -e^{-x}$$

Now, let $y = \sin h\, x = \frac{1}{2}(e^x - e^{-x})$

Then,
$$\frac{dy}{dx} = \frac{1}{2}\left(\frac{d}{dx}(e^x) - \frac{d}{dx}(e^{-x})\right)$$

$$= \frac{1}{2}[e^x - (-e^{-x})] = \frac{1}{2}(e^x + e^{-x}) = \cos h\, x.$$

X. $\boldsymbol{\frac{d}{dx}(\cos h\, x) = \sin h\, x}$

Proof is similar to that of (IX).

XI. $\boldsymbol{\frac{d}{dx}(\tan h\, x) = \sec h^2 x}$

Proof: Let,
$$y = \tan h\, x = \frac{\sin h\, x}{\cos h\, x}$$

$$\frac{dy}{dx} = \frac{\frac{d}{dx}(\sin h\, x) \cos h\, x - \sin h\, x \frac{d}{dx}(\cos h\, x)}{(\cos h\ x)^2}$$

$$= \frac{(\cos\ h\, x)(\cos\ h\, x) - (\sin\ h\, x)(\sin\ h\, x)}{\cos\ h^2\ x}$$

$$= \frac{\cos h^2 x - \sin h^2 x}{\cos h^2 x} = \frac{1}{\cos h^2 x} = \sec\ h^2\, x.$$

XII. $\dfrac{d}{dx}(\sec h\, x) = -\sec h\, x \tan h\, x$

NOTES

Proof: Let,
$$y = \sec h\, x = \frac{1}{\cos h\, x}$$
$$\frac{dy}{dx} = \frac{\frac{d}{dx}(1)\cos h\, x - (1)\frac{d}{dx}(\cos h\, x)}{\cos h^2\, x}$$
$$= \frac{(0)(\cos h\, x) - \sin h\, x}{\cos h^2\, x}$$
$$= -\left(\frac{\sin h\, x}{\cos h\, x}\right)\left(\frac{1}{\cos h\, x}\right) = -\tan h\, x \sec h\, x.$$

Example 2.13: If $y = x^2 \sin x$, find $\dfrac{dy}{dx}$.

Solution: This is a problem of the type $\dfrac{d}{dx}(uv)$.

By applying the formula,
$$\frac{dy}{dx} = \frac{d}{dx}(x^2)\sin x + x^2\frac{d}{dx}(\sin x)$$
$$= 2x \sin x + x^2 \cos x.$$

Example 2.14: If $y = x^2 \operatorname{cosec} x$, find $\dfrac{dy}{dx}$.

Solution: We can write y as $\dfrac{x^2}{\sin x}$

Applying the formula,
$$y = \frac{x^2}{\sin x}$$
$$\Rightarrow \qquad \frac{dy}{dx} = \frac{\frac{d}{dx}(x^2)\sin x - x^2\frac{d}{dx}(\sin x)}{\sin^2 x}$$
$$= \frac{2x\sin x - x^2\cos x}{\sin^2 x}.$$

2.4.3 Chain Rule of Differentiation

This is the most important and widely used rule for differentiation.

The rule states that:

If y is a differentiable function of z, and z is a differentiable function of x, then y is a differentiable function of x, i.e.,
$$\frac{dy}{dx} = \frac{dy}{dz}\cdot\frac{dz}{dx}.$$

Proof: Let $y = F(z)$ and $z = f(x)$.

If δx is change in x and corresponding changes in y and z are δy and δz respectively, then $y + \delta y = F(z + \delta z)$ and $z + \delta z = f(x + \delta x)$.

Thus, $\qquad \delta y = F(z + \delta z) - F(z)$ and $\delta z = f(x + \delta x) - f(x)$

NOTES

Now, $$\frac{\delta y}{\delta x} = \frac{\delta y}{\delta z} \cdot \frac{\delta z}{\delta x}$$

$$\Rightarrow \qquad \lim_{\delta x \to 0} \frac{\delta y}{\delta x} = \lim_{\delta x \to 0} \frac{\delta y}{\delta z} \lim_{\delta x \to 0} \frac{\delta z}{\delta x}$$

$$\Rightarrow \qquad \frac{dy}{dx} = \left(\lim_{\delta z \to 0} \frac{\delta y}{\delta z} \right) \frac{dz}{dx},$$

Since $\delta x \to 0$ implies that $\delta z \to 0$

$$= \frac{dy}{dz} \cdot \frac{dz}{dx}.$$

Corollary: If y is a differentiable function of x_1, x_1 is a differentiable function of x_2, ... , x_{n-1} is a differentiable function of x, then y is a differentiable function of x.

And $$\frac{dy}{dx} = \frac{dy}{dx_1} \frac{dx_1}{dx_2} \cdots \frac{dx_{n-1}}{dx_n}.$$

Proof: Apply induction on n.

Example 2.15: Find the differential coefficient of sin log x with respect to x.

Solution: Put $z = \log x$, then, $y = \sin z$

Now, $$\frac{dy}{dx} = \frac{dy}{dz} \cdot \frac{dz}{dx} = \cos z \cdot \frac{1}{x} = \frac{1}{x} \cos (\log x).$$

Example 2.16: Find the differential coefficient of (*i*) $e^{\sin x^2}$ (*ii*) $\log \sin x^2$ with respect to x.

Solution: (*i*) Put $x^2 = y$, $\sin x^2 = z$ and $u = e^{\sin x^2}$

Then, $u = e^z$, $z = \sin y$ and $y = x^2$

By chain rule,

$$\frac{du}{dx} = \frac{du}{dz} \frac{dz}{dy} \frac{dy}{dx}$$

$$= e^z \cos y \, 2x = e^{\sin y} \cos y \, 2x = 2xe^{\sin x^2} \cos x^2.$$

(*ii*) Let, $$u = x^2$$

$$v = \sin x^2 = \sin u$$

Then, $$y = \log \sin x^2 = \log \sin u = \log v$$

So, $$\frac{du}{dx} = 2x, \quad \frac{dv}{du} = \cos u \text{ and } \frac{dy}{dv} = \frac{1}{v}$$

Then, $$\frac{dy}{dx} = \frac{dy}{dv} \cdot \frac{dv}{du} \cdot \frac{du}{dx}$$

$$= \frac{1}{v} \cdot \cos u \cdot 2x$$

$$= \frac{1}{\sin u} \cos u \cdot 2x = 2x \cot u = 2x \cot x^2.$$

Note: After some practice we can use the chain rule, without actually going through the substitutions. For example,

If $y = \log (\sin x^2)$, then $\dfrac{dy}{dx} = \dfrac{1}{\sin x^2} \cos x^2 \cdot 2x = 2x \cot x^2$.

NOTES

Note that we have first differentiated log function according to the formula $\frac{d}{dt}(\log t) = \frac{1}{t}$. Since, here we have log (sin x^2), so the first term on differentiation is $\frac{1}{\sin x^2}$.

Now, consider $\sin x^2$ and differentiate it according to the formula $\frac{d}{du}(\sin u) = \cos u$. Thus, the second term is $\cos x^2$.

Finally, we differentiated x^2 with respect to x, so, the third term is $2x$.

Then, we multiplied all these three terms to get the answer $2x \cot x^2$. We will illustrate this quick method by few more examples.

Example 2.17: Find $\frac{dy}{dx}$, when $y = e^{(2x+3)^3}$.

Solution: Since, $\frac{d(e^t)}{dt} = e^t$ and $\frac{du^3}{du} = 3u^2$

And $\frac{d(2v)}{dv} = 2$

We get, $\frac{dy}{dx} = e^{(2x+3)^2} \cdot 3(2x+3)^2 \cdot (2.1+0) = 6(2x+3)^2 e^{(2x+3)^2}$

Example 2.18: Differentiate $y = \log[\sin(3x^2 + 5)]$ with respect to x.

Solution: $\frac{dy}{dx} = \frac{1}{\sin(3x^2+5)} \cos(3x^2+5)\, 6x = 6x \cot(3x^2 + 5)$.

Example 2.19: Differentiate $y = \tan^2(\sqrt{x} + 3)$.

Solution: $\frac{dy}{dx} = 2\tan(\sqrt{x}+3) \cdot \sec^2(\sqrt{x}+3) \cdot \frac{1}{2\sqrt{x}} = \frac{\tan(\sqrt{x}+3)\sec^2(\sqrt{x}+3)}{\sqrt{x}}$.

CHECK YOUR PROGRESS

1. Find out the limit of $\frac{x^2-1}{x-1}$ as $x \to 1$.
2. Find $\frac{dy}{dx}$ and $\left(\frac{dy}{dx}\right)_{x=3}$ for $y = x^3$.
3. When is a function $f(x)$ differentiable at $x = a$?
4. Differentiate following two functions with respect to x:
   $3x^2 - 6x + 1$ and $(2x^2 + 5x - 7)^{5/2}$
5. How will you calculate the derivative of a product of two functions?
6. What do you understand by the chain rule of differentiation?

2.5 DERIVATIVES: TANGENT AND NORMAL

NOTES

Figure 2.1 shows the tangent and normal of function $y = f(x)$.

Let $P\ (x', y')$ be any point on the curve $y = f(x)$. Equation of any line through (x', y') is

$$y - y' = m(x - x')$$

In case this line is tangent at P, its slope must be equal to the value of $\frac{dy}{dx}$ at (x', y').

Let us denote the value of $\frac{dy}{dx}$ at (x', y') by $\left(\frac{dy}{dx}\right)'$

Thus, $$m = \left(\frac{dy}{dx}\right)'$$

Hence, the equation of tangent at (x', y') of $y = f(x)$ is given by

$$y - y' = \left(\frac{dy}{dx}\right)'(x - x')$$

Sometimes this formula is written as

$$Y - y' = \left(\frac{dy}{dx}\right)(X - x),$$

Where, X and Y are current coordinates and (x, y) is the given point (i.e., the point of contact).

A normal at a point P of a plane curve $y = f(x)$ is a line through P in the plane of curve, perpendicular to the tangent there at P.

So, if PT is tangent to a curve $y = f(x)$ at a point P, and if PN is perpendicular to PT, then PN is normal to the curve at P.

The equation of normal at $P\ (x', y')$ is, therefore,

$$y - y' = \frac{1}{\left(\frac{dy}{dx}\right)'}(x - x')$$

NOTES

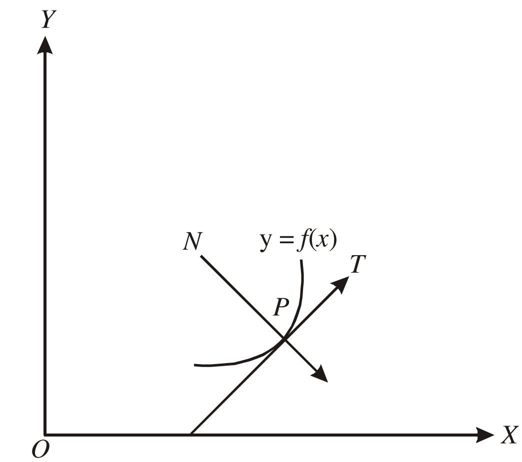


Figure 2.1 *Tangent and Normal of y = f(x)*

Or, $$y - y' \frac{1}{\left(\frac{dy}{dx}\right)'}(x - x') = 0$$

***Note*:** As in the case of tangent, if we take X and Y to be current coordinates and P as the point (x, y) the equation of normal at P can be written as

$$(Y - y')\left(\frac{dy}{dx}\right) + (X - x) = 0$$

Example 2.20: Find the equation tangent at $(x - y')$ on the ellipse

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

Solution: $$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

$$\Rightarrow \quad \frac{2x}{a^2} + \frac{2y}{b^2} \cdot \frac{dy}{dx} = 0$$

$$\Rightarrow \quad \frac{dy}{dx} = -\frac{b^2 x}{a^2 y}$$

$$\Rightarrow \quad \left(\frac{dy}{dx}\right)' = -\frac{b^2 x'}{a^2 y'}$$

The equation of tangent at (x', y') is therefore,

$$y - y' = -\left(\frac{dy}{dx}\right)' . (x - x')$$

Or, $$y - y' = -\frac{b^2 x'}{a^2 y'}(x - x')$$

$$\Rightarrow \quad a^2 yy' - a^2 y'^2 = -b^2 xx' + b^2 x'^2$$

$$\Rightarrow \quad b^2 xx' - a^2 yy' = b^2 x'^2 + a^2 y'^2$$

$$\Rightarrow \quad \frac{xx'}{a^2} + \frac{yy'}{b^2} = \frac{x'^2}{a^2} + \frac{y'^2}{b^2}$$

$$= 1, \text{ since } (x' - y') \text{ lies on the ellipse.}$$

Example 2.21: Find the equation of normal at a point $(x' - y')$ on the curve $y = a \log \sin x$.

Solution: Here, $$\frac{dy}{dx} = a\frac{1}{\sin x}\cos x = a \cot x$$

$$\Rightarrow \quad \left(\frac{dy}{dx}\right)' = a \cot x'$$

So, the normal has equation

$$(y - y')\, a \cos x' + (x - x') = 0$$

i.e.,

$$a(y - y')\, a \cos x' + (x - x') \sin x^2 = 0$$

Example 2.22: Find the equation of tangent at a point (2, 2) on the curve $y^2 = 2x$.

Solution: Differentiating $y^2 = 2x$ with respect to x, we get

$$2y\frac{dy}{dx} = 2$$

$$\Rightarrow \quad \frac{dy}{dx} = \frac{1}{y}$$

So, $\frac{dy}{dx}$ at the point (2, 2) is equal to $\frac{1}{2}$.

i.e., $$\left(\frac{dy}{dx}\right)' = \frac{1}{2}$$

Hence, the equation of tangent to $y^2 = 2x$ at (2, 2) is

$$y - y' = \left(\frac{dy}{dx}\right)'(x - x')$$

i.e., $$y - 2 = \frac{1}{2}(x - 2)$$

Or, $$2y - 4 = (x - 2)$$

$$\Rightarrow \quad x - 2y + 2 = 0$$

Example 2.23: Find the equation of normal at a point $\left(3, \frac{4}{3}\right)$ on $xy = 4$.

NOTES

Solution: Differentiating $xy = 4$ with respect to x, we get $y + x\frac{dy}{dx} = 0$

$$\Rightarrow \qquad \frac{dy}{dx} = -\frac{y}{x}$$

So, $\quad \frac{dy}{dx}$ at $\left(3, \frac{4}{3}\right)$ is equal to $= \frac{-\frac{4}{3}}{3} = -\frac{4}{9}$.

Hence, the equation of normal at $\left(3, \frac{4}{3}\right)$ is

$$\left(y - \frac{4}{3}\right)\left(-\frac{4}{9}\right) + (x - 3) = 0$$

Or, $$\left(-4y + \frac{16}{3}\right) + 9(x - 3) = 0$$

Or, $$-12y + 16 + 27x - 81 = 0$$

Or, $$27x - 12y - 65 = 0.$$

Subtangent and Subnormal

Figure 2.2. shows the subtangent and subnormal of $y = f(x)$. As shown in the figure, let PT and PN respectively be the tangent and the normal at a point P of the curve $y = f(x)$.

PM is perpendicular from P on x-axis. Let the tangent and normal at P meet the x-axis at K and L respectively.

Length PK is called the length of tangent at P and length PL is called the length of normal at P.

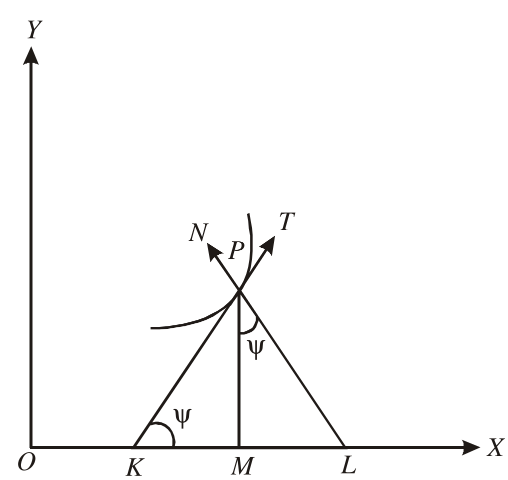


Figure 2.2 *Subtangent and Subnormal of y = y(x)*

Further, *KM,* i.e., the projection of *KP* on *x*-axis is called *subtangent* at *P* and *LM*, the projection of *PL* on *x*-axis is called the *subnormal* at *P*.

Suppose that (x', y') are the coordinates of *P*, then if ψ is the angle *TKL*,

NOTES

$$\left(\frac{dy}{dx}\right) = \tan\psi$$

$$\angle MPL = \frac{\pi}{2} - \angle MPL = \frac{\pi}{2} - \left(\frac{\pi}{2} - \psi\right) = \psi$$

So, subtangent at $P = KM = PM\ \dfrac{KM}{PM} = y' \cot\psi$

$$= y' \frac{1}{\left(\frac{dy}{dx}\right)'}$$

$$= \frac{y'}{\left(\frac{dy}{dx}\right)'}$$

Whereas, subnormal at $P = LM = PM.\ \dfrac{LM}{PM} = y^2 \tan\psi$

$$= y' \left(\frac{dy}{dx}\right)'$$

The length of tangent at P = PK will be

$$= \sqrt{PM^2 + KM^2}$$

$$= \sqrt{y'^2 + \left[\frac{y'}{\left({dy}/{dx}\right)'}\right]^2}$$

$$= \frac{y}{\left(\frac{dy}{dx}\right)'}\sqrt{\left[\left(\frac{dy}{dx}\right)'\right]^2 + 1}$$

$$= \frac{y'}{\left(\frac{dy}{dx}\right)'}\sqrt{1 + \left[\left(\frac{dy}{dx}\right)'\right]^2}$$

Also, the length of normal at P = PL will be

$$= \sqrt{PM^2 + LM^2}$$

$$= \sqrt{y'^2 + \left[y'\left(\frac{dy}{dx}\right)'\right]^2}$$

$$= y' \sqrt{1 + \left[\left(\frac{dy}{dx}\right)'\right]^2}$$

Example 2.24: Prove that for the parabola $y^2 = 4ax$, subtangent at any point is twice its abscissa and subnormal is constant.

Solution: Let (x', y') be any point on $y'^2 = 4ax$.

NOTES

So, $$y'^2 = 4ax' \qquad ...(1)$$

Differentiating, we get

$$2y\frac{dy}{dx} = 4a \Rightarrow \frac{dy}{dx} = \frac{2a}{y}$$

$$\Rightarrow \qquad \left(\frac{dy}{dx}\right)' = \frac{2a}{y'}$$

Hence, the subtangent at (x', y') is

$$\frac{y'}{\left(\frac{dy}{dx}\right)'} = \frac{y'}{\left(\frac{2a}{y'}\right)} = \frac{y'}{2a}$$

$$= \frac{4ax'}{2a}, \text{ by Equation (1)}$$

$$= 2x'$$

$$= \text{twice the abscissae of the point } P.$$

Also, the subnormal at P is equal to $y'\left(\frac{dy}{dx}\right)'$

$$= y'.\frac{2a}{y'} = 2a, \text{ a constant.}$$

Example 2.25: Find the lengths of tangent and normal at any point (x', y') of the curve $y = \frac{a}{2}\left(e^{x/a} + e^{-x/a}\right)$.

Solution: Here, $$y = \frac{a}{2}\left(e^{x/a} + e^{-x/a}\right)$$

$$= a\cosh\frac{x}{a}$$

$$\Rightarrow \qquad \frac{dy}{dx} = a\left(\sin h\frac{x}{a}\right)\frac{1}{a}$$

$$= \sinh\frac{x}{a}$$

This gives that $$\left(\frac{dy}{dx}\right)' = \sinh\frac{x'}{a}$$

Then, the length of the tangent at (x', y') is $\dfrac{y'}{\left(\dfrac{dy}{dx}\right)'}\sqrt{1+\left[\left(\dfrac{dy}{dx}\right)'\right]^2}$

$$= \frac{a\cos h\dfrac{x'}{a}\sqrt{1+\sin h^2\dfrac{x'}{a}}}{\sin h\dfrac{x'}{a}}$$

As (x', y') lies on the given curve.

$$= \frac{a\cos h\dfrac{x'}{a}}{\sin h\dfrac{x'}{a}}\cos h\frac{x'}{a}$$

$$= a \cos h\frac{x'}{a} \qquad \cot h\frac{x'}{a}$$

Also, the length of normal at (x', y') is $y'\sqrt{1+\left[\left(\dfrac{dy}{dx}\right)'\right]^2}$

$$= \left(a \cos h\frac{x'}{a}\right)\sqrt{1+\sin h^2\frac{x'}{a}}$$

$$= a \cos h^2\frac{x'}{a}$$

2.6 DIFFERENTIATION OF IMPLICIT FUNCTIONS AND PARAMETRIC FORMS

2.6.1 Parametric Differentiation

When x and y are separately given as functions of a single variable t (called a *parameter*), then you should first evaluate $\dfrac{dx}{dt}$ and $\dfrac{dy}{dt}$ and then use chain formula $\dfrac{dy}{dt} = \dfrac{dy}{dx}\dfrac{dx}{dt}$, to obtain

$$\frac{dy}{dx} = \frac{\dfrac{dy}{dt}}{\dfrac{dx}{dt}}$$

The equations $x = F(t)$ and $y = G(t)$ are called *parametric equations.*

Example 2.26: Let $x = a\left(\cos t + \log \tan \frac{t}{2}\right)$ and $y = a \sin t$, find $\frac{dy}{dx}$.

NOTES

Solution:
$$\frac{dx}{dt} = a\left(-\sin t + \frac{1}{\tan t/2}\sec^2 \frac{t}{2}\cdot\frac{1}{2}\right)$$
$$= a\left(-\sin t + \frac{1}{2\sin t/2 \cos t/2}\right)$$
$$= a\left(-\sin t + \frac{1}{\sin t}\right)$$
$$= \frac{a(1-\sin^2 t)}{\sin t} = \frac{a\cos^2 t}{\sin t} \text{ and } \frac{dy}{dt} = a\cos t$$

Hence,
$$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{a\cos t}{(a\cos^2 t/\sin t)} = \frac{\sin t}{\cos t} = \tan\ t.$$

Example 2.27: Determine $\frac{dy}{dx}$, where $x=a\,(1 + \sin\theta)$ and $y = a(1 - \cos\theta)$.

Solution:
$$\frac{dx}{d\theta} = a\,(1 + \cos\theta) = 2a\cos^2\frac{\theta}{2}$$

Also,
$$\frac{dy}{d\theta} = a\sin\theta = 2a\sin\frac{\theta}{2}\cos\frac{\theta}{2}$$

So,
$$\frac{dy}{dx} = \frac{\frac{dy}{d\theta}}{\frac{dx}{d\theta}} = \frac{\sin\theta/2}{\cos\theta/2} = \tan\frac{\theta}{2}$$

2.6.2 Logarithmic Differentiation

Whenever you have a function which is either a product or quotient of functions whose differential coefficients are known or whose variables occur in powers, you take the help of logarithms to differentiate. This makes the task of finding differential coefficients much easier than with the usual method. This technique is illustrated below with the help of examples.

Example 2.28: Prove that $\frac{d}{dx}(x^x) = x^x(1 + \log x)$

Solution: Let, $y = x^x$

Then, $\log y = x \log x$

$$\Rightarrow \quad \frac{d}{dx}(\log y) = (1)\log x + x\left(\frac{1}{x}\right) = 1 + \log x$$

$$\Rightarrow \quad \frac{d}{dy}(\log y)\frac{dy}{dx} = 1 + \log x$$

$$\Rightarrow \quad \frac{1}{y}\frac{dy}{dx} = 1 + \log x$$

$$\Rightarrow \qquad \frac{dy}{dx} = y(1 + \log x) = x^x(1 + \log x)$$

Note: Since $\log e = 1$, the above result can also be written as,

$$\frac{d}{dx}(x^x) = x^x(\log ex).$$

NOTES

Example 2.29:

(*i*) Differentiate $y = x^2(x + 1)(x^3 + 3x + 1)$ with respect to x.

(*ii*) If $x^m y^n = (x + y)^{m+n}$, prove that $\frac{dy}{dx} = \frac{y}{x}$.

Solution:

(*i*) $$y = x^2(x + 1)(x^3 + 3x + 1)$$

$$\Rightarrow \quad \log y = 2\log x + \log(x + 1) + \log(x^3 + 3x + 1)$$

$$\Rightarrow \quad \frac{1}{y}\frac{dy}{dx} = \frac{2}{x} + \frac{1}{x+1} + \frac{3x^2+3}{x^3+3x+1}$$

$$\Rightarrow \quad \frac{dy}{dx} = y\left(\frac{2}{x} + \frac{1}{x+1} + \frac{3x^2+3}{x^3+3x+1}\right)$$

$$= x^2(x+1)(x^3+3x+1)\left[\frac{2}{x} + \frac{1}{x+1} + \frac{3(x^2+1)}{x^3+3x+1}\right]$$

(*ii*) $$x^m y^n = (x + y)^{m+n}$$

$$\Rightarrow \quad m\log x + n\log y = (m + n)\log(x + y)$$

$$\Rightarrow \quad \frac{m}{x} + \frac{n}{y}\frac{dy}{dx} = \left(\frac{m+n}{x+y}\right)\cdot\left(1 + \frac{dy}{dx}\right)$$

$$\Rightarrow \quad \left(-\frac{m+n}{x+y} + \frac{n}{y}\right)\frac{dy}{dx} = \frac{m+n}{x+y} - \frac{m}{x}$$

$$\Rightarrow \quad \left[\frac{-my - ny + nx + ny}{y(x+y)}\right]\frac{dy}{dx} = \frac{mx + nx - mx - my}{x(x+y)}$$

$$\Rightarrow \quad \left[\frac{(-my + nx)}{y(x+y)}\right]\frac{dy}{dx} = \frac{nx - my}{x(x+y)}$$

$$\Rightarrow \quad \frac{dy}{dx} = \frac{y}{x}.$$

Example 2.30:

(*i*) If $x^y = e^{x-y}$, show that

$$\frac{dy}{dx} = \frac{\log x}{(1 + \log x)^2}$$

(*ii*) Find $\frac{dy}{dx}$ if $y = x^{e^x}$.

Solution:

(*i*) Given, $$x^y = e^{x-y}$$

$$\Rightarrow \quad y\log x = (x - y)\log e = x - y$$

NOTES

$$\Rightarrow \qquad y(1 + \log x) = x$$

$$\Rightarrow \qquad y = \frac{x}{1+\log x}$$

$$\Rightarrow \qquad \frac{dy}{dx} = \frac{(1)(1+\log x) - x\left(\frac{1}{x}\right)}{(1+\log x)^2}$$

$$= \frac{1+\log x - 1}{(1+\log x)^2} = \frac{\log x}{(1+\log x)^2}$$

(*ii*) Given $\qquad y = x^{e^x}$

$$\log y = e^x \log x$$

$$\Rightarrow \qquad \frac{1}{y}\frac{dy}{dx} = e^x \log x + e^x \ \frac{1}{x}$$

$$\Rightarrow \qquad \frac{dy}{dx} = y\left(e^x \log x + \frac{e^x}{x}\right)$$

$$= x^{e^x}\left(e^x \log x + \frac{e^x}{x}\right)$$

$$= x^{e^x} e^x \log x + e^x x^{(e^x - 1)}$$

Differentiation of One Function with Respect to Another Function and the Substitution Method

Parametric differentiation is also applied in differentiating one function with respect to another function, x being treated as a parameter. Sometimes a proper substitution makes the solution of such problems quite easy.

Example 2.31:

(*i*) Differentiate x with respect to x^3.

(*ii*) Differentiate $\tan^{-1}\frac{2x}{1-x^2}$ with respect to x.

Solution:

(*i*) Let $y = x$ and $z = x^3$

We have to evaluate $\frac{dy}{dz}$.

Now, $\qquad \frac{dy}{dx} = 1 \quad \text{and} \quad \frac{dz}{dx} = 3x^2$

So, $\qquad \frac{dy}{dz} = \frac{\frac{dy}{dx}}{\frac{dz}{dx}} = \frac{1}{3x^2}$

(*ii*) Let, $\qquad y = \tan^{-1}\left(\frac{2x}{1-x^2}\right)$

Putting, $\qquad x = \tan\theta$, we find that

$$y = \tan^{-1}\left(\frac{2\tan\theta}{1-\tan^2\theta}\right)$$

$$= \tan^{-1}(\tan 2\theta) = 2\theta = 2\tan^{-1}x$$

So, $$\frac{dy}{dx} = 2\frac{1}{1+x^2} = \frac{2}{1+x^2}.$$

Example 2.32: Differentiate sin x with respect to log x.

Solution: Let $y = \sin x$ and $z = \log x$

Then, $$\frac{dy}{dx} = \cos x \quad \text{and} \quad \frac{dz}{dx} = \frac{1}{x}$$

Imply that, $$\frac{dy}{dz} = \frac{\frac{dy}{dx}}{\frac{dz}{dx}} = \frac{\cos x}{\frac{1}{x}} = x\cos x.$$

Example 2.33: Differentiate $\tan^{-1}\left(\frac{\sqrt{1+x^2}-1}{x}\right)$ with respect to $\tan^{-1}x$.

Solution: Let, $$y = \tan^{-1}\left(\frac{\sqrt{1+x^2}-1}{x}\right) \text{ and } z = \tan^{-1}x$$

Then,

$$\frac{dy}{dx} = \frac{1}{1+\left(\frac{\sqrt{1+x^2}-1}{x}\right)^2}\cdot\frac{\left(\frac{1}{2\sqrt{1+x^2}}2x\right)x-(\sqrt{1+x^2}-1)}{x^2}$$

$$= \frac{x^2}{x^2+1+x^2+1-2\sqrt{1+x^2}}\cdot\frac{x^2-(1+x^2-\sqrt{1+x^2})}{x^2\sqrt{1+x^2}}$$

$$= \frac{1}{2(1+x^2-\sqrt{1+x^2})}\frac{\sqrt{1+x^2}-1}{\sqrt{1+x^2}}$$

$$= \frac{1}{2\sqrt{1+x^2}(\sqrt{1+x^2}-1)}\frac{\sqrt{1+x^2}-1}{\sqrt{1+x^2}} = \frac{1}{2(1+x^2)}$$

And $$\frac{dz}{dx} = \frac{1}{1+x^2}$$

So, $$\frac{dy}{dz} = \frac{\frac{dy}{dx}}{\frac{dz}{dx}} = \frac{1}{2}$$

Aliter: $z = \tan^{-1}x \Rightarrow x = \tan z$

NOTES

So,
$$y = \tan^{-1}\left(\frac{\sqrt{1+\tan^2 z}-1}{\tan z}\right)$$
$$= \tan^{-1}\left[\frac{\sec z - 1}{\tan z}\right]$$
$$= \tan^{-1}\left[\frac{1-\cos z}{\sin z}\right]$$
$$= \tan^{-1}\left[\frac{2\sin^2 z/2}{2\sin z/2 \cos z/2}\right] = \tan^{-1}(\tan z/2) = z/2$$

So,
$$\frac{dy}{dz} = \frac{1}{2}$$

Differentiation '*ab initio*' or by First Principle

Earlier we discussed how to differentiate some standard functions starting from the definition. Here, we have more examples to illustrate the techniques.

Example 2.34: Differentiate $\sqrt{\cos x}$ with respect to x by first principle.

Solution: Let $y = \sqrt{\cos x}$.

If δx changes in x, then the corresponding change δy in y is given by
$$y + \delta y = \sqrt{\cos(x+\delta x)}$$

So,
$$\delta y = \sqrt{\cos(x+\delta x)} - \sqrt{\cos x}$$

Or,
$$\frac{\delta y}{\delta x} = \frac{\sqrt{\cos(x+\delta x)} - \sqrt{\cos x}}{\delta x}$$
$$= \frac{\cos(x+\delta x) - \cos x}{\delta x[\sqrt{\cos(x+\delta x)} + \sqrt{\cos x}]}$$
$$= \frac{2\sin\left(\frac{-\delta x}{2}\right)\sin\left(x+\frac{\delta x}{2}\right)}{\delta x[\sqrt{\cos(x+\delta x)} + \sqrt{\cos x}]}$$

Thus,
$$\frac{dy}{dx} = \lim_{\delta x \to 0}\frac{\delta y}{\delta x} = \lim_{\delta x\to 0}\left(\frac{-2\sin\frac{\delta x}{2}}{\delta x}\right)\cdot\frac{\sin x}{\sqrt{\cos x}+\sqrt{\cos x}}$$
$$= -\frac{\sin x}{2\sqrt{\cos x}}\lim_{\delta x\to 0}\left(\frac{\sin\frac{\delta x}{2}}{\frac{\delta x}{2}}\right)$$
$$= -\frac{\sin x}{2\sqrt{\cos x}} \text{ as } \lim_{\delta x\to 0}\left(\frac{\sin\frac{\delta x}{2}}{\frac{\delta x}{2}}\right) = 1.$$

Example 2.35: Differentiate $e^{\sqrt{x}}$ ab initio.

Solution: Let $y = e^{\sqrt{x}}$ and let δx be the change in x, corresponding to which δy is the change in y.

NOTES

Then, $y + \delta y = e^{\sqrt{x+\delta x}}$

$\Rightarrow \quad \delta y = e^{\sqrt{x+\delta x}} - e^{\sqrt{x}}$

$$\Rightarrow \quad \frac{\delta y}{\delta x} = \frac{e^{\sqrt{x}}\,(e^{\sqrt{x+\delta x}-\sqrt{x}} - 1)}{\delta x}$$

$$= e^{\sqrt{x}}\left(\frac{e^{\sqrt{x+\delta x}-\sqrt{x}} - 1}{\sqrt{x+\delta x}-\sqrt{x}}\right)\left(\frac{\sqrt{x+\delta x}-\sqrt{x}}{\delta x}\right)$$

$$= e^{\sqrt{x}}\,\frac{\left[1+(\sqrt{x+\delta x}-\sqrt{x})+\frac{(\sqrt{x+\delta x}-\sqrt{x})^2}{2!}+\ldots-1\right]}{\sqrt{x+\delta x}-\sqrt{x}}\left[\frac{\sqrt{x+\delta x}-\sqrt{x}}{\delta x}\right]$$

$$= e^{\sqrt{x}}\left[1+\frac{(\sqrt{x+\delta x}-\sqrt{x})}{2!}+\ldots\right]\frac{\sqrt{x}\left[\left(1+\frac{\delta x}{x}\right)^{1/2}-1\right]}{\delta x}$$

$$\Rightarrow \lim_{\delta x\to 0}\frac{\delta y}{\delta x} = (e^{\sqrt{x}}\sqrt{x})\lim_{\delta x\to 0}\left[\frac{1+\frac{1}{2}\frac{\delta x}{x}+\frac{\frac{1}{2}\left(\frac{1}{2}-1\right)}{2!}\left(\frac{\delta x}{x}\right)^2+\ldots-1}{\delta x}\right]$$

$$= (e^{\sqrt{x}}\sqrt{x})\lim_{\delta x\to 0}\left(\frac{1}{2}\frac{1}{x}-\frac{1}{8}\frac{\delta x}{x^2}+\ldots\right)$$

$$= \frac{e^{\sqrt{x}}\sqrt{x}}{2x} = \frac{1}{2}\frac{e^{\sqrt{x}}}{\sqrt{x}}$$

Example 2.36: Starting from definition find the derivative of tan $(2x + 3)$.

Solution: Let $y = \tan(2x + 3)$

Then, $y + \delta y = \tan(2x + 2\delta x + 3)$

$\Rightarrow \quad \delta y = \tan(2x + 2\delta x + 3) - \tan(2x + 3)$

$$= \frac{\sin(2x+2\delta x+3)}{\cos(2x+2\delta x+3)} - \frac{\sin(2x+3)}{\cos(2x+3)}$$

$$= \frac{\sin(2x+2\delta x+3)\cos(2x+3)-\sin(2x+3)\cos(2x+2\delta x+3)}{\cos(2x+2\delta x+3)\cos(2x+3)}$$

$$= \frac{\sin(2\delta x)}{\cos(2x+2\delta x+3)\cos(2x+3)}$$

NOTES

$$\Rightarrow \qquad \frac{\delta y}{\delta x} = \frac{\sin(2\delta x)}{\delta x} \cdot \frac{1}{\cos(2x+3)\cos(2x+2\delta x+3)}$$

$$\Rightarrow \qquad \frac{\delta y}{\delta x} = 2\left[\lim_{\delta x \to 0} \frac{\sin(2\delta x)}{2\delta x}\right] \frac{1}{\cos^2(2x+3)}$$

$$= 2\sec^2(2x+3) \text{ as } \lim_{\delta x \to 0} \frac{\sin(2\delta x)}{2\delta x} = 1.$$

2.6.3 Successive Differentiation

Let $y = f(x)$, then $\frac{dy}{dx}$ is again a function, say, $g(x)$ of x. We can find $\frac{dg(x)}{dx}$. This is called *second deriative* of y with respect to x and is denoted by $\frac{d^2y}{dx^2}$ or by y_2.

In similar fashion we can define $\frac{d^3y}{dx^3}, \frac{d^4y}{dx^4}, ..., \frac{d^ny}{dx^n}$ for any positive integer n.

Note: Sometimes $y^{(n)}$ or $D^n(y)$ are also used in place of $\frac{d^ny}{dx^n}$ or y_n.

The process of differentiating a function more than once is called successive differentiation.

Example 2.37: Differentiate $x^3 + 5x^2 - 7x + 2$ four times.

Solution: Let, $y = x^3 + 5x^2 - 7x + 2$

then, $y_1 = 3x^2 + 10x - 7$

$y_2 = 6x + 10$

$y_3 = 6$ and $y_4 = 0$.

Some Standard Formulas for the *n*th Derivative

I. $\boldsymbol{y = (ax + b)^m}$

Here, $y_1 = m(ax+b)^{m-1}a = ma(ax+b)^{m-1}$

$y_2 = ma(m-1)(ax+b)^{m-2}a$

So, $y_2 = m(m-1)a^2(ax+b)^{m-2}$

Thus, $y_3 = m(m-1)a^2(m-2)(ax+b)^{m-3}a$

$= m(m-1)(m-2)a^3(ax+b)^{m-3}$

Proceeding in this manner, we find that

$$y_n = m(m-1)(m-2)\ \dots\ (m-n+1)a^n(ax+b)^{m-n}$$

Aliter: The above result can also be obtained by the principle of Mathematical Induction. The result has already been proved true for $n = 1$.

Suppose it is true for $n = k$,

i.e., $y_k = m(m-1)(m-2)\ \dots\ (m-k+1)\ a^k(ax+b)^{m-k}$

Differentiating once more with respect to x, we get

$$y_{k+1} = m(m-1)(m-2)\ \dots\ (m-k+1)\ a^k (m-k)(ax+b)^{m-k-1}a$$
$$= m(m-1)(m-2)\ \dots\ (m-k+1)\ [m-(k+1)+1]a^{k+1}$$
$$(ax+b)^{m-(k+1)}$$

Hence, the result is true for $n = k + 1$ also. Consequently, the formula holds true for all positive integral values of n.

***Corollary* 1:** If $y = x^m$, then

$$y_n = m(m-1)\ \dots\ (m-n+1)x^{m-n}.$$

***Corollary* 2:** If $y = x^m$ and m is a positive integer, then

$$y_m = m(m-1)\ \dots\ (m-m+1)x^0 = m!$$

And $\quad y_{m+1} = 0, \quad y_n = 0 \ \forall\, n > m.$

***Corollary* 3:** If $y = (ax+b)^{-1}$, then

$$y_n = (-1)(-2)\ \dots\ (-1-n+1)\ a^n(ax+b)^{-1-n}$$
$$\Rightarrow \quad y_n = (-1)^n\ n!\ a^n(ax+b)^{-(n+1)}.$$

II. $\quad \boldsymbol{y = \sin(ax+b)}$

Here, $\quad y_1 = a\cos(ax+b)$

$$= a\sin\left(ax+b+\frac{\pi}{2}\right) \qquad \left[\because \sin\left(\theta+\frac{\pi}{2}\right) = \cos\theta\right]$$

$$y_2 = a^2\cos\left(ax+b+\frac{\pi}{2}\right)$$
$$= a^2\sin\left(ax+b+\frac{\pi}{2}+\frac{\pi}{2}\right) = a^2\sin\left(ax+b+2\,\frac{\pi}{2}\right)$$
$$y_3 = a^3\cos\left(ax+b+\frac{2\pi}{2}\right)$$
$$= a^3\sin\left(ax+b+\frac{2\pi}{2}+\frac{\pi}{2}\right) = a^3\sin\left(ax+b+\frac{3\pi}{2}\right)$$

Proceeding in this manner, we get

$$y_n = a^n\sin\left(ax+b+\frac{n\pi}{2}\right).$$

Note: All the formulas discussed above can be proved by using the principle of Mathematical Induction. We have illustrated the technique in alternative method of formula (I).

***Corollary*:** For $y = \cos(ax+b)$

$$y_n = a^n\cos\left(ax+b+\frac{n\pi}{2}\right).$$

Proof: $\quad y = \cos(ax+b) = \sin\left(ax+b+\frac{\pi}{2}\right)$

So, $\quad y_n = a^n\sin\left(ax+b+\frac{\pi}{2}+\frac{n\pi}{2}\right) = a^n\cos\left(ax+b+\frac{n\pi}{2}\right).$

NOTES

III. $\boldsymbol{y = e^{ax}}$

Clearly, $y_1 = ae^{ax}$

$y_2 = a^2e^{ax}$

$y_3 = a^3e^{ax}$ ... and so on, till we get

$y_n = a^ne^{ax}$.

IV. $\boldsymbol{y = \log (ax + b)}$

Here, $y_1 = \dfrac{a}{ax+b} = a\,(ax + b)^{-1}$

$$y_n = \frac{d^{n-1}}{dx^{n-1}}(y_1)$$

$$= a(-1)^{n-1}(n-1)!a^{n-1}(ax+b)^{-1-(n-1)}$$

By Corollary (3) and (I)

$$= (-1)^{n-1}(n-1)!\; a^n(ax+b)^{-n}$$

$$= \frac{(-1)^{n-1}(n-1)!a^n}{(ax+b)^n}.$$

V. $\boldsymbol{y = e^{ax}\cos(bx + c)}$

In this case,

$$y_1 = ae^{ax}\cos(bx+c) - be^{ax}\sin(bx+c)$$

$$= e^{ax}[\gamma\cos\varphi\cos(bx+c) - \gamma\sin\varphi\sin(bx+c)]$$

Where, $a = \gamma\cos\varphi$ and $b = \gamma\sin\varphi$

So, $y_1 = \gamma\, e^{ax}\cos(bx+c+\varphi)$

Again, $y_2 = \gamma\,[ae^{ax}\cos(bx+c+\varphi) - be^{ax}\sin(bx+c+\varphi)]$

$$= \gamma^2e^{ax}[\cos\varphi\cos(bx+c+\varphi) - \sin\varphi\sin(bx+c+\varphi)]$$

$$= \gamma^2e^{ax}\cos(bx+c+2\varphi)$$

Proceeding in this manner, we get

$$y_n = \gamma^ne^{ax}\cos(bx+c+n\varphi)$$

Where, $\tan\varphi = \dfrac{b}{a}$ and $\gamma = (a^2+b^2)^{1/2}$

$[\because\ a = \gamma\cos\varphi,\ b = \gamma\sin\varphi \Rightarrow \dfrac{b}{a} = \tan\varphi$ and $a^2 + b^2 = \gamma^2]$.

***Corollary*:** For $y = e^{ax}\sin(bx+c)$

$$y_n = \gamma^ne^{ax}\sin(bx+c+n\varphi)$$

Where, $\varphi = \tan^{-1}\dfrac{b}{a}$ and $\gamma = (a^2+b^2)^{1/2}$

Proof is left as an exercise.

VI. $\boldsymbol{y = \tan^{-1}\left(\dfrac{x}{a}\right)}$

Now, $y_1 = \dfrac{1}{1+\dfrac{x^2}{a^2}}\cdot\dfrac{1}{a}$

$$= \frac{a}{a^2 + x^2}$$

$$= \frac{a}{(x+ia)(x-ia)}, \text{ where } i = \sqrt{-1}$$

$$= \frac{1}{2i}\left[\frac{1}{x-ia} - \frac{1}{x+ia}\right]$$

$$= \frac{1}{2i}[(x-ia)^{-1} - (x+ia)^{-1}]$$

$$\Rightarrow \qquad y_n = D^{n-1}(y_1)$$

$$= \frac{1}{2i}\Big[(n-1)!\,(-1)^{n-1}(x-ia)^{-1-(n-1)} - (n-1)!\,(-1)^{n-1}(x+ia)^{-1-(n-1)}\Big]$$

$$= \frac{(-1)^{n-1}(n-1)!}{2i}[(x-ia)^{-n} - (x+ia)^{-n}]$$

Put, $\qquad x = \gamma \cos\theta$ and $a = \gamma \sin\theta$

Then, $\qquad \tan\theta = \frac{a}{x}$ and $\gamma = \frac{a}{\sin\theta}$

Thus, $$y_n = \frac{(-1)^{n-1}(n-1)!}{2i}[\gamma^{-n}(\cos\theta - i\sin\theta)^{-n} - \gamma^{-n}(\cos\theta - i\sin\theta)^{-n}]$$

$$= \frac{(-1)^{n-1}(n-1)!}{2i\gamma^n}[\cos n\theta + i\sin n\theta - (\cos n\theta - i\sin n\theta)]$$

[By De Moivre's theorem $(\cos\theta + i\sin\theta)^n = \cos n\theta + i\sin n\theta$ for an integer n.]

$$= \frac{(-1)^{n-1}(n-1)!\,2i\sin n\theta}{2i\gamma^n}$$

$$= \frac{(-1)^{n-1}(n-1)!\sin n\theta}{a^n/\sin^n\theta}$$

$$= \frac{(-1)^{n-1}(n-1)!\sin n\theta \cdot \sin^n\theta}{a^n}$$

Where, $\qquad \theta = \tan^{-1}\left(\frac{a}{x}\right)$

Note: Since $\tan\theta = \frac{a}{x} \Rightarrow \frac{x}{a} = \cot\theta = \tan y$

We get, $\qquad \theta = \frac{\pi}{2} - y$, so the above formula can also be put in the form

$$y_n = \frac{(-1)^{n-1}(n-1)!\sin n\left(\frac{\pi}{2} - y\right)\sin^n\left(\frac{\pi}{2} - y\right)}{a^n}$$

NOTES

To prove De Moivre's theorem for an integer we proceed as:

For $n = 1$, $(\cos\theta + i\sin\theta)^1 = \cos\theta + i\sin\theta = \cos 1\theta + i\sin 1\theta$

For $n = 2$, $(\cos\theta + i\sin\theta)^2 = \cos^2\theta - \sin^2\theta + 2i\sin\theta\cos\theta$

$$= \cos 2\theta + i\sin 2\theta$$

Proceeding in this manner, we get $(\cos\theta + i\sin\theta)^n = \cos n\theta + i\sin n\theta$

In case n is negative integer, put $n = -m$, $m > 0$

$$(\cos\theta + i\sin\theta)^n = \frac{1}{(\cos\theta + i\sin\theta)^m} = \frac{1}{\cos m\theta + i\sin m\theta}$$

$$= \frac{\cos m\theta - i\sin m\theta}{\cos^2 m\theta + \sin^2 m\theta} = \cos m\theta - i\sin m\theta$$

$$= \cos(-m)\theta + i\sin(-m)\theta = \cos n\theta + i\sin n\theta$$

Note: By $y_n(a)$ we shall mean the value of y_n at $x = a$.

Thus, for example, if $y = \sin 3x$

$$y_4\left(\frac{\pi}{3}\right) = \left(3^4 \sin\left(3x + \frac{4\pi}{2}\right)\right) \text{ at } x = \frac{\pi}{3}$$

$$= 81 \sin(3x + 2\pi) \quad \text{at} \quad x = \frac{\pi}{3}$$

$$= 81 \sin 3x \quad \text{at } x = \frac{\pi}{3}$$

$$= 0$$

2.7 PARTIAL DIFFERENTIATION

Till now we have been talking about functions of one variable. But there may be functions of more than one variable. For example,

$$z = \frac{xy}{x+y},\ u = x^2 + y^2 + z^2$$

are functions of two and three variables, respectively. Another example is, demand for any good depends not only on the price of the goods, but also on the income of the individuals and on the price of related goods.

Let $z = f(x, y)$ be function of two variables x and y. x and y can take any value independent of each other. If you allot a fixed value to one variable, say x, and second variable y is allowed to vary, then $f(x, y)$ can be regarded as a function of single variable y. So, you can talk of its derivative with respect to y, in the usual sense. We call this partial derivative of z with respect to y, and it is denoted by the symbol $\frac{\partial z}{\partial y}$.

Thus, we have

$$\frac{\partial z}{\partial y} = \lim_{\delta y \to 0} \frac{f(x, y + \delta y) - f(x, y)}{\delta y}$$

Similarly, we define partial derivative of z with respect to x, as the derivative of z, regarded as a function of x alone. Thus, here y is kept constant and x is allowed to vary.

So, $$\frac{\partial z}{\partial x} = \lim_{\delta x \to 0} \frac{f(x+\delta x, y) - f(x, y)}{\delta x}.$$

Note: $\frac{\partial z}{\partial x}$ is also denoted by z_x and

$\frac{\partial z}{\partial y}$, by z_y.

In similar manner, we can define $\frac{\partial^2 z}{\partial x^2}, \frac{\partial^2 z}{\partial x \partial y}, \frac{\partial^2 z}{\partial y \partial x}, \frac{\partial^2 z}{\partial y^2}$. Thus, $\frac{\partial^2 z}{\partial x^2}$ is nothing but $\frac{\partial}{\partial x}\left(\frac{\partial z}{\partial x}\right)$; $\frac{\partial^2 z}{\partial x \partial y}$ is same as $\frac{\partial}{\partial x}\left(\frac{\partial z}{\partial y}\right)$, $\frac{\partial^2 z}{\partial y \partial x} = \frac{\partial}{\partial y}\left(\frac{\partial z}{\partial x}\right)$ and $\frac{\partial^2 z}{\partial y^2} = \frac{\partial}{\partial y}\left(\frac{\partial z}{\partial y}\right)$. In this manner one can define partial derivatives of higher orders.

Note: In general $\frac{\partial^2 z}{\partial x \partial y} \neq \frac{\partial^2 z}{\partial y \partial x}$, i.e., change of order of differentiation does not always yield the same answer. There are famous theorems like Young's theorem and Schwarz theorem which give sufficient conditions for two derivatives to be equal. But as far as we are concerned, all the functions that we deal with in this book are supposed to satisfy the relation $\frac{\partial^2 z}{\partial x \partial y} = \frac{\partial^2 z}{\partial y \partial x}$.

Example 2.38: Evaluate $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$.

When, $$z = \frac{x^2}{x-y+1}$$

Solution: $$z = \frac{x^2}{x-y+1}$$

$$\frac{\partial z}{\partial x} = \frac{2x(x-y+1) - x^2 \frac{\partial}{\partial x}(x-y+1)}{(x-y+1)^2}$$

$$= \frac{2x^2 - 2xy + 2x - x^2(1)}{(x-y+1)^2}$$

$$= \frac{x^2 - 2xy + 2x}{(x-y+1)^2} = \frac{x(x-2y+2)}{(x-y+1)^2}$$

Again, $$\frac{\partial z}{\partial y} = \frac{\partial}{\partial y}\left(\frac{x^2}{x-y+1}\right)$$

$$= x^2 \frac{\partial}{\partial y}[(x-y+1)^{-1}] = x^2\left[-(x-y+1)^{-2}\frac{\partial}{\partial y}(-y)\right]$$

$$= -x^2(x-y+1)^{-2}(-1) = \frac{x^2}{(x-y+1)^2}$$

Example 2.39: Verify that for $z = \log (x^2 - y^2)$.

$$\frac{\partial^2 z}{\partial x \partial y} = \frac{\partial^2 z}{\partial y \partial x}.$$

NOTES

Solution:
$$\frac{\partial z}{\partial x} = \frac{\partial}{\partial x}[\log (x^2 - y^2)] = \frac{2x}{x^2 - y^2}$$

$$\frac{\partial^2 z}{\partial y \partial x} = \frac{\partial}{\partial y}\left(\frac{2x}{x^2 - y^2}\right) = 2x\frac{\partial}{\partial y}[(x^2 - y^2)^{-1}]$$

$$= 2x\left[-(x^2 - y^2)^{-2}\frac{\partial}{\partial y}(-y^2)\right]$$

$$= -2x(x^2 - y^2)^{-2}(-2y) = \frac{+4xy}{(x^2 - y^2)^2} \quad ...(1)$$

Further,
$$\frac{\partial z}{\partial y} = \frac{\partial}{\partial y}[\log (x^2 - y^2)] = \frac{-2y}{x^2 - y^2}$$

$$\frac{\partial^2 z}{\partial x \partial y} = \frac{\partial}{\partial x}\left[-\frac{2y}{x^2 - y^2}\right] = (-2y)\frac{\partial}{\partial x}[(x^2 - y^2)^{-1}]$$

$$= (-2y)\left[-(x^2 - y^2)^{-2}\frac{\partial}{\partial x}(x^2)\right] = \frac{4xy}{(x^2 - y^2)^2} \quad ...(2)$$

Equations (1) and (2) give the required result.

Example 2.40: Show that $\frac{\partial^2}{\partial x \partial y}[(x + y)e^{x+y}] = (x + y + 2)e^{x+y}$.

Solution:
$$\frac{\partial}{\partial y}\left[(x + y)e^{x+y}\right] = \frac{\partial}{\partial y}[e^x e^y (x + y)]$$

$$= e^x[e^y(x + y) + e^y]$$

$$= e^x e^y(x + y + 1)$$

$$\frac{\partial^2}{\partial x \partial y}\left[(x + y)e^{x+y}\right] = \frac{\partial}{\partial x}[e^x e^y(x + y + 1)]$$

$$= e^y \frac{\partial}{\partial x}[e^x(x + y + 1)]$$

$$= e^y[e^x(x + y + 1) + e^x]$$

$$= e^x e^y(x + y + 1 + 1)$$

$$= e^{x+y}(x + y + 2)$$

$$= (x + y + 2)\, e^{x+y}$$

Example 2.41: If $u = \log (x^2 + y^2 + z^2)$, prove that:

$$x\frac{\partial^2 u}{\partial y \partial z} = y\frac{\partial^2 u}{\partial z \partial x} = z\frac{\partial^2 u}{\partial x \partial y}.$$

Solution:
$$u = \log (x^2 + y^2 + z^2)$$

NOTES

$$\Rightarrow \qquad \frac{\partial u}{\partial x} = \frac{d}{d(x^2+y^2+z^2)}[\log(x^2+y^2+z^2)]\frac{\partial(x^2+y^2+z^2)}{\partial x}$$

$$= \frac{1}{x^2+y^2+z^2}\cdot 2x = \frac{2x}{x^2+y^2+z^2}$$

$$\Rightarrow \qquad \frac{\partial^2 u}{\partial z \partial x} = \frac{\partial}{\partial z}\left(\frac{\partial u}{\partial x}\right) = -\frac{2x}{(x^2+y^2+z^2)^2}.(2z) = \frac{-4xz}{(x^2+y^2+z^2)^2}$$

$$\Rightarrow \qquad y\frac{\partial^2 u}{\partial z \partial x} = \frac{-4xyz}{(x^2+y^2+z^2)^2} \qquad ...(1)$$

Again, $$\frac{\partial^2 u}{\partial x \partial y} = \frac{\partial^2 u}{\partial y \partial x} = \frac{\partial}{\partial y}\left[\frac{\partial u}{\partial x}\right]$$

$$= -\frac{2x}{(x^2+y^2+z^2)^2}(2y) = \frac{-4xy}{(x^2+y^2+z^2)^2}$$

$$\Rightarrow \qquad z\frac{\partial^2 u}{\partial x \partial y} = -\frac{4xyz}{(x^2+y^2+z^2)^2} \qquad ...(2)$$

Similarly, it can be shown that

$$x\frac{\partial^2 u}{\partial y \partial z} = -\frac{4xyz}{(x^2+y^2+z^2)^2} \qquad ...(3)$$

Equations (1), (2) and (3) give the required result.

Example 2.42: If $u = f(r)$, where $r = \sqrt{x^2+y^2}$,

prove that $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = f''(r) + \frac{1}{r}f'(r)$.

Solution: $r = (x^2+y^2)^{1/2}$

$$\Rightarrow \qquad \frac{\partial r}{\partial x} = \frac{1}{2}\cdot(x^2+y^2)^{-1/2}\cdot 2x = \frac{x}{\sqrt{x^2+y^2}} = \frac{x}{r}$$

Similarly, $$\frac{\partial r}{\partial y} = \frac{y}{r}.$$

Now, $u = f(r)$

$$\Rightarrow \qquad \frac{\partial u}{\partial x} = \frac{d}{dr}(u)\frac{\partial r}{\partial x}$$

$$\Rightarrow \qquad = \frac{u'x}{r} \text{ where } u' = \frac{du}{dr} = f'(r)$$

$$\Rightarrow \qquad \frac{\partial^2 u}{\partial x^2} = \frac{u'}{r} + \frac{x}{r}\frac{\partial u'}{\partial x} + u'x\frac{\partial}{\partial x}\left(\frac{1}{r}\right)$$

$$= \frac{u'}{r} + \frac{x}{r}\frac{d}{dr}(u')\frac{\partial r}{\partial x} + xu'\frac{d}{dx}\left(\frac{1}{r}\right)\frac{\partial r}{\partial x}$$

NOTES

$$= \frac{u'}{r} + \frac{x^2}{r^2}u'' - \frac{xu'}{r^2}\frac{x}{r} \text{ where } u'' = \frac{d^2u}{dr^2} = f''(r)$$

$$= \frac{u'}{r} + \frac{x^2u''}{r^2} - \frac{x^2}{r^3}u'.$$

In a similar manner, it can be proved that:

$$\frac{\partial^2 u}{\partial y^2} = \frac{u'}{r} + \frac{y^2u''}{r^2} - \frac{y^2}{r^3}u'$$

So, $$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = \frac{2u'}{r} + \frac{u''}{r^2}(x^2+y^2) - \frac{u'}{r^3}(x^2+y^2)$$

$$= \frac{2u'}{r} + \frac{u''}{r^2}\cdot r^2 - \frac{u'}{r^3}\cdot r^2 = \frac{2u'}{r} + u'' - \frac{u'}{r}$$

$$= u'' + \frac{u'}{r} = f''(r) + \frac{1}{r}f'(r).$$

2.8 MAXIMA AND MINIMA OF FUNCTIONS

Definition 1

The point $(c, f(c))$ is called a *maximum point* of $y = f(x)$, if (*i*) $f(c+h) \le f(c)$, and (*ii*) $f(c-h) \le f(c)$ for small $h \ge 0$. $f(c)$ itself is called a *maximum value* of $f(x)$.

Definition 2

The point $(d, f(d))$ is called a *minimum point* of $y = f(x)$, if

(*i*) $f(d+h) \ge f(d)$, and

(*ii*) $f(d-h) \ge f(d)$

for all small $h \ge 0$.

$f(d)$ itself is called a *minimum value* of $f(x)$.

Thus, you observe that points $P[c-h, f(c-h)]$ and $Q[c+h, f(c+h)]$, which are very near to A, have ordinates less than that of A, whereas the points

$R[d-h, f(d-h)]$, and $S[d+h, f(d+h)]$,

Which are very close to B, have ordinates greater than that of B.

Figure 2.3 exhibits these maximum and minima points.

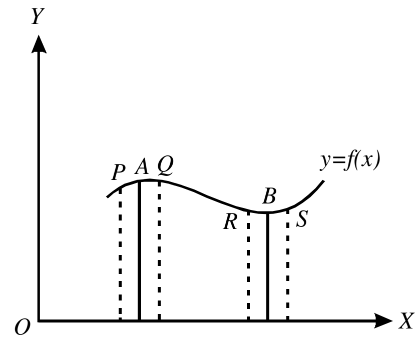


Figure 2.3 *Maxima and Minima*

NOTES

We will now prove that at a maximum or minimum point, the first differential coefficient with respect to x must vanish (in other words, tangents at a maximum or minimum point is parallel to x-axis, which is, otherwise, evident from Figure 2.3).

Let $[c, f(c)]$ be a maximum point and let $h \geq 0$ be a small number.

Since $\quad f(c-h) \leq f(c)$

We have, $\quad f(c-h) - f(c) \leq 0$

$$\Rightarrow \quad \frac{f(c-h) - f(c)}{-h} \geq 0 \qquad \text{... (2.3)}$$

Again, $\quad f(c+h) \leq f(c) \Rightarrow f(c+h) - f(c) \leq 0$

$$\Rightarrow \quad \frac{f(c+h) - f(c)}{h} \leq 0 \qquad \text{... (2.4)}$$

Equation (2.3) implies that $\displaystyle \lim_{k \to 0} \frac{f(c+k) - f(c)}{k} \geq 0,$ $\quad$ [put $k = -h$]

and equation (2.4) gives that $\displaystyle \lim_{k \to 0} \frac{f(c+h) - f(c)}{k} \leq 0$ $\quad$ [put $k = h$]

Thus, $\quad \displaystyle 0 \leq \lim_{k \to 0} \frac{f(c+k) - f(c)}{k} \leq 0$

$\Rightarrow \quad \left(\dfrac{dy}{dx}\right)$ at $x = c$ is equal to zero.

i.e., $\quad f'(c) = 0$

Again, let $[d, f(\text{d})]$ be a minimum point and let $h \geq 0$ be a small number.

Since $\quad f(d-h) \geq f(d)$

we have, $\quad f(d-h) - f(d) \geq 0$

NOTES

$$\Rightarrow \qquad \frac{f(d-h)-f(d)}{-h} \le 0 \qquad \text{... (2.5)}$$

Again, $\quad f(d+h) \ge f(d)$

$$\Rightarrow \qquad \frac{f(d+h)-f(d)}{h} \ge 0 \qquad \text{... (2.6)}$$

Equations (2.5) and (2.6) imply $\displaystyle \lim_{k\to 0}\frac{f(d+k)-f(d)}{k} = 0$

i.e., $\quad f'(d) = 0$

Before we proceed to find out the criterion for determining whether a point is maximum or minimum, we will discuss the increasing and decreasing functions of x.

A function $f(x)$ is said to be *increasing* (*decreasing*) if

$$f(x+c) \ge f(x) \ge f(x-c)\ [f(x+c) \le f(x) \le f(x-c)] \text{ for all } c \ge 0.$$

Theorem 2.1

If $f'(x) \ge 0$, then $f(x)$ is increasing function of x and if $f'(x) \le 0$, then $f(x)$ is decreasing function of x.

Proof: $\quad f'(x) \ge 0 \quad \Rightarrow \displaystyle \lim_{\delta x\to 0}\frac{f(x+\delta x)-f(x)}{\delta x} \ge 0 \qquad \text{... (2.7)}$

In case $\delta x > 0$, put $c = \delta x$, then equation (2.7) gives

$$f(x+c) \ge f(x)$$

In case $\delta x < 0$, put $c = -\delta x$, then equation (2.7) gives

$$\frac{f(x-c)-f(x)}{-c} \ge 0$$

$$\Rightarrow \qquad f(x-c) - f(x) \le 0$$

$$\Rightarrow \qquad f(x) \ge f(x-c)$$

Hence, $\quad f(x+c) \ge f(x) \ge f(x-c)$

In other words, $f(x)$ is increasing function of x.

Suppose that $f'(x) \le 0$

Then, $\quad \displaystyle \lim_{\delta x\to 0}\frac{f(x+\delta x)-f(x)}{\delta x} \le 0 \qquad \text{... (2.8)}$

In case $\delta x > 0$, put $c = \delta x$, then Equation (2.8) gives

$$f(x+c) - f(c) \le 0$$

i.e., $\quad f(x+c) \le f(x)$

If $\delta x < 0$, put $c = -\delta x$, then Equation (2.8) gives

$$\frac{f(x-c) - f(x)}{-c} \leq 0$$

$$\Rightarrow \qquad f(x-c) - f(x) \geq 0$$

$$\Rightarrow \qquad f(x) \leq f(x-c)$$

So, $\qquad f(x+c) \leq f(x) \leq f(x-c)$

This means that $f(x)$ is a decreasing function of x.

NOTES

Notes:

1. A function $f(x)$ is said to be strictly increasing (strictly decreasing) if
   $f(x+c) > f(x) > f(x-c)$ $[f(x+c) < f(x) < f(x-c)]$ for all $c > 0$.
2. It is seen that $f(x)$ is increasing, if $f(x) > f(y)$, whenever $x > y$, and $f(x)$ is decreasing, if
   $x > y \Rightarrow \ f(x) < f(y)$ and conversely.
3. It can be proved as above that a function $f(x)$ is strictly increasing or strictly decreasing accordingly, if
   $f'(x) > 0$ or $f'(x) < 0$.

Geometrically, the above theorem means that for an increasing function, tangent at any point makes acute angle with OX whereas for a decreasing function, tangent at any point makes an obtuse angle with x-axis. This is shown in Figure 2.4.

Let A be a maximum point $(c, f(c))$ of a curve $y = f(x)$.

Let $P\ [c-h, f(c-h)]$ and $Q\ [c+h, f(c+h)]$ be two points in the vicinity of A (i.e., h is very small).

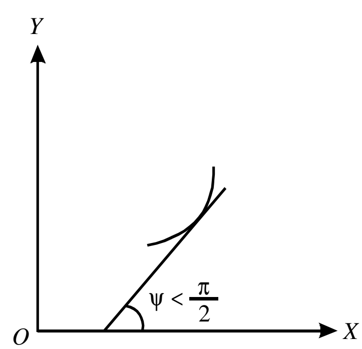


Figure 2.4 *Tangent for an Increasing Function Making an Acute Angle with X-axis*

Figure 2.5(a) shows the tangent of the function making one obtuse angle with Y-axis. If ψ_1 and ψ_2 are inclinations of tangents at P and Q respectively, it is quite obvious from the Figure 2.5(b), that ψ_1 is acute and ψ_2 is obtuse.

Analytically, it is apparent from the fact that function is increasing from P to A and decreasing from A to Q. So, $\tan\psi$ decreases as we pass through A ($\tan\psi$ is +ve when ψ is acute and it is –ve when ψ is obtuse).

NOTES

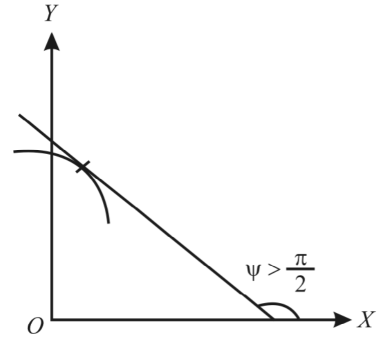


Figure 2.5(a) *Tangent of the function making an Obtuse Angle with Y-axis*

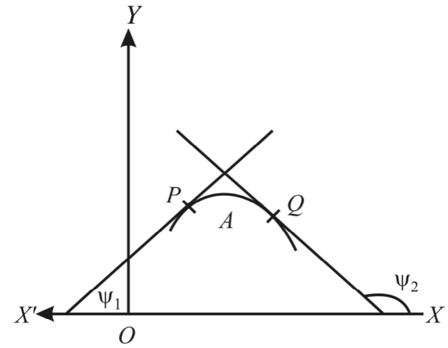


Figure 2.5(b) *Inclinations of Tangents at P and Q*

Thus, $\frac{dy}{dx} = \tan\psi$ is a decreasing function of x. In other words, $\frac{d^2y}{dx^2} \leq 0$.
Since $\tan\psi$ is strictly decreasing function of x, ($f(x)$ is not a constant function), so, $\frac{d^2y}{dx^2} < 0$. Consequently, at a maximum point $c\,(f(c))$,

$$f''(c) < 0.$$

Similarly, it can be easily seen that if $R\,[d-h, f(d-h)]$ and $S\,[d+h, f(d+h)]$ are two points in the neighbourhood of a minimum point $B[d, f(d)]$, slopes of tangents as we pass through B increase. (Here ψ_1 is obtuse, so $\tan\psi_1 < 0$ and ψ_2 is acute, so $\tan\psi_2 > 0$).

Therefore, for a minimum point $(d, f(d))$, $\frac{d^2y}{dx^2} > 0$, i.e., $f''(d) > 0$.
Figure 2.6 shows the slopes of the tangent passing through B.

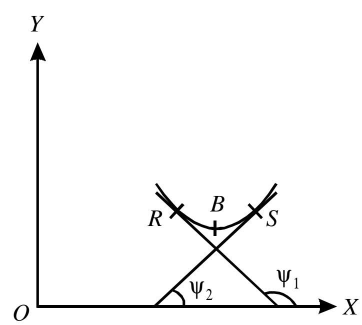


Figure 2.6 *Slopes of the Tangent Passing through B*

Notes:

1. A point (α, β), such that $f'(\alpha) = 0, f''(\alpha) \neq 0$ and $f'''(\alpha) \neq 0$ is called a *point of inflexion.*

NOTES

2. Any point at which $\frac{dy}{dx} = 0$ is called a *stationary point*. Thus, maxima and minima are stationary points. A stationary point need not be a maximum or a minimum point (it could be a point of flexion). Value of $f(x)$ at a stationary point is called *stationary value*.

   We have the following rule for the determination of maxima and minima, if they exist, of a function $y = f(x)$.

Step I. Putting $\frac{dy}{dx} = 0,$ calculate the stationary points.

Step II. Compute $\frac{d^2y}{dx^2}$ at these stationary points.

In case $\frac{d^2y}{dx^2} > 0,$ the stationary point is a minimum point.

In case $\frac{d^2y}{dx^2} < 0,$ the stationary point is a maximum point.

If $\frac{d^2y}{dx^2} = 0,$ then compute $\frac{d^3y}{dx^3}.$

If $\frac{d^3y}{dx^3} \neq 0,$ the stationary point is neither a maximum nor a minmum at that point.

If $\frac{d^3y}{dx^3} = 0,$ find $\frac{d^4y}{dx^4}.$ If the fourth derivative is negative at that point, then there is a maximum and if it is positive then there is a minimum.

Again in case $\frac{d^4y}{dx^4} = 0,$ find the fifth derivative and proceed as above till we get a definite answer.

Example 2.43: Find the maximum and minimum values of the expression

$$x^3 - 3x^2 - 9x + 27.$$

Solution: Let $y = x^3 - 3x^2 - 9x + 27$

$$\frac{dy}{dx} = 3x^2 - 6x - 9$$

For maxima and minima, $\frac{dy}{dx} = 0$

$$\Rightarrow \quad 3x^2 - 6x - 9 = 0$$

$$\Rightarrow \quad (x - 3)(x + 1) = 0$$

$$\Rightarrow \quad x = -1, 3$$

Now, $\frac{d^2y}{dx^2} = 6x - 6$

At $x = -1$, $\frac{d^2 y}{dx^2} = -12 < 0$, so $x = -1$ gives a maximum point of y.

Again, at $x = 3$, $\frac{d^2 y}{dx^2} = +12 > 0, x = 3$ gives a minimum point of y.

NOTES

Hence, maximum value of y is $[(-1)^3 - 3\,(-1)^2 - 9\,(-1) + 27]$

$= 36 + 1 - 3 = 34$

While minimum value of y is $3^3 - 3(3)^2 - 9\,(3) + 27$

$= 54 - 27 - 27 = 0$

Example 2.44: Find the maximum and minimum values of the function

$$8x^5 - 15x^4 + 10x^2.$$

Solution: Let $f(x) = 8x^5 - 15x^4 + 10x^2$

$\Rightarrow \quad f'(x) = 40x^4 - 60x^3 + 20x$

For maxima and minima,

$$f'(x) = 0 \quad \Rightarrow \quad x = 0,\ 1,\ -\frac{1}{2}$$

So, these are the points where there can be a maximum or a minimum

Now, $f''(x) = 160x^3 - 180x^2 + 20$

Thus, $f''(0) = 20 > 0 \quad \Rightarrow \quad$ There is a minimum at $x = 0$

Again, $\quad f''(-\frac{1}{2}) = 160\,(-\frac{1}{2})^3 - 180\,(-\frac{1}{2})^2 + 20 = -45 < 0$

$\Rightarrow \quad$ there is a maximum at $x = -\frac{1}{2}$

Since $f''(1) = 160 - 180 + 20 = 0$, we cannot say anything regarding a maximum or a minimum at $x = 1$ at this stage. So, we find $f'''(x)$.

Now, $f'''(x) = 480\,x^2 - 360\,x$

But, $\quad f'''(1) = 480 - 360 \neq 0$

$\Rightarrow \quad$ There is neither a maximum nor a minimum at $x = 1$

Hence, $f\,(-\frac{1}{2}) = \frac{21}{16}$ is maximum value and $f\,(0) = 0$ is minimum value.

Example 2.45: Find out maxima and minima of $\sin x + \cos x$, when x lies between 0 and 2π.

Solution: Let $y = \sin x + \cos x, \quad 0 \leq x \leq 2\pi$

For maxima and minima, $\frac{dy}{dx} = 0$

$\Rightarrow \quad \cos x - \sin x = 0$

$\Rightarrow \quad \tan x = 1$

$\Rightarrow \quad x = \frac{\pi}{4} \quad \text{or} \quad \frac{3\pi}{4}.$

Now, $\frac{d^2y}{dx^2} = -\sin x - \cos x$

At, $x = \frac{\pi}{4}, \frac{d^2y}{dx^2} = -\frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}} = -\sqrt{2} < 0$

And at, $x = \frac{3\pi}{4}, \frac{d^2y}{dx^2} = \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}} = \sqrt{2} > 0$

So, $x = \frac{\pi}{4}$ gives a maximum and $x = \frac{3\pi}{4}$ gives a minimum point of given function.

NOTES

Example 2.46: Find maxima and minima of

$\sin x + \cos 2x \quad \text{for } 0 \le x \le \pi/2.$

Solution: Let $y = \sin x + \cos 2x$

For maxima and minima $\frac{dy}{dx} = 0$

$\Rightarrow \quad \cos x - 2 \sin 2x = 0$

$\Rightarrow \quad \cos x(1 - 4 \sin x) = 0$

$\Rightarrow \quad \cos x = 0 \text{ or } \sin x = \frac{1}{4}$

$\Rightarrow \quad x = \frac{\pi}{2} \text{ or } x = \sin^{-1}\frac{1}{4}$

Now, $\frac{d^2y}{dx^2} = -\sin x - 4 \cos 2x$

$= -\sin x - 4(1 - 2\sin^2 x)$

$= 8\sin^2 x - \sin x - 4$

At, $x = \pi/2, \frac{d^2y}{dx^2} = 8 - 1 - 4 = 3 > 0$

So, $x = \frac{\pi}{2}$ gives a minimum point of $\sin x + \cos 2x$

At, $x = \sin^{-1}\frac{1}{4}, \frac{d^2y}{dx^2} = 8\left(\frac{1}{16}\right) - \frac{1}{4} - 4$

$= \frac{1}{2} - \frac{1}{4} - 4$

$$= \frac{1}{2} - \frac{17}{4} = -\frac{15}{4} < 0$$

NOTES

Consequently, $x = \sin^{-1}\frac{1}{4}$ gives a maximum point of $\sin x + \cos 2x$.

Example 2.47: The sum of two numbers is 24. Find the numbers if the sum of their squares is to be minimum.

Solution: Let x and y be two numbers such that

$$x + y = 24$$

Let, $s = x^2 + y^2 = x^2 + (24 - x)^2$

For maxima and minima, $\frac{ds}{dx} = 0$

$\Rightarrow \quad 2x - 2\,(24 - x) = 0$

$\Rightarrow \quad 2x = 24 \quad \Rightarrow \quad x = 12$

Further, $\frac{d^2s}{dx^2} = 4 > 0$

So, $x = 12$ and $y = 12$ give minimum value.

Hence, the required numbers are 12 and 12.

CHECK YOUR PROGRESS

7. What is parametric differentiation?
8. Determine $\frac{dy}{dx}$ where $x = a\,(1 + \sin\theta)$ and $y = a\,(1 - \cos\theta)$.
9. When do we take help of logarithmic differentiation?
10. What is successive differentiation?
11. Find first four derivatives of $y = x^3 + 5x^2 - 7x + 2$.
12. What do you understand by partial differentiation?
13. When is a point called maximum and has a maximum value?
14. When is a function $f(x)$ said to be increasing or decreasing?

2.9 SUMMARY

In this unit, you have learned that:

- In mathematics, usually there are two kinds of quantities: constants and variables.
- A quantity which is liable to vary is called a variable quantity or simply a variable, and a quantity that retains its value through all mathematical operations is termed as a constant quantity or a constant.

- Limits of a function can be evaluated using the expansion method.
- The derivative of the sum (or difference) of two functions is equal to the sum (or difference) of their derivatives.
- The derivative of a product of two functions = (the derivative of first function × second function) + (first function × derivative of second function).
- The derivative of a constant function is equal to the constant multiplied by the derivative of the function.
- The chain rule of differentiation states that if y is a differentiable function of z, and z is a differentiable function of x, then y is a differentiable function of x.
- Logarithms are used to find differential coefficients of functions which are either product or quotient of functions whose differential coefficients are known, or whose variables occur in powers.
- Parametric differentiation is also applied in differentiating one function with respect to another function, x being treated as a parameter.

NOTES

2.10 KEY TERMS

- **Derivative:** It refers to the instantaneous rate of change of a function.
- **Variable:** It refers to a quantity that is liable to vary.
- **Constant:** It refers to a quantity that retains its value through all mathematical operations.
- **Differentiation:** It refers to the rate of change of any quantity with respect to the change in another quantity with which it has a functional relationship.
- **Maxima:** It refers to the largest value that a function takes in a point either within a given neighbourhood or on the function domain in its entirety.
- **Minima:** It refers to the smallest value that a function takes in a point either within a given neighbourhood or on the function domain in its entirety.

2.11 ANSWERS TO 'CHECK YOUR PROGRESS'

1. $\displaystyle \lim_{x \to 1} \frac{x^2 - 1}{x - 1} = 2.$
2. 27.
3. A function $f(x)$ is said to be derivable or differentiable at $x = a$ if its derivative exists at $x = a$.

NOTES

4. $6x - 6$ and $\frac{5}{2}(4x+5)(2x^2+5x-7)^{3/2}$.

5. The derivative of a product of two functions = (the derivative of first function × second function) + (first function × derivative of second function).

6. The chain rule is the most important and widely used rule for differentiation. The rule states that if y is a differentiable function of z, and z is a differentiable function of x, then y is a differentiable function of x.

7. When x and y are separately given as functions of a single variable t (called a parameter), then we first evaluate $\frac{dx}{dt}$ and $\frac{dy}{dt}$ and then use chain formula $\frac{dy}{dt} = \frac{dy}{dx}\frac{dx}{dt}$, to obtain $\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}}$

   The equations $x = F(t)$ and $y = G(t)$ are called parametric equations.

8. $$\frac{dy}{dx} = \frac{\frac{dy}{d\theta}}{\frac{dx}{d\theta}} = \frac{\sin \theta/2}{\cos \theta/2} = \tan\frac{\theta}{2}.$$

9. Whenever we have a function which is a product or quotient of functions whose differential coefficients are known or a function, in which variables occur in powers, we take the help of logarithms.

10. The process of differentiating a function more than once is called successive differentiation.

11. Let $y = z^3 + 5x^2 - 7x + 2$

    then, $y_1 = 3x^2 + 10x - 7$

    $y_2 = 6x + 10$

    $y_3 = 6$ and

    $y_4 = 0$.

12. Let $z = f(x, y)$ be function of two variables x and y. x and y can take any value independently of each other. If we allot a fixed value to one variable, say x, and second variable y is allowed to vary, $f(x, y)$ can be regarded as a function of single variable y. So we can talk of its derivative with respect to y, in the usual sense. We call this partial derivative of z with respect to y, and denote it by the symbol $\frac{\partial z}{\partial y}$.

Thus, we have,

$$\frac{\partial z}{\partial y} = \lim_{\delta y \to 0} \frac{f(x, y+\delta y) - f(x, y)}{\delta y}$$

Similarly, we define partial derivative of z with respect to x, as the derivative of z, regarded as a function of x alone. Thus, here y is kept constant and x is allowed to vary.

So, $\dfrac{\partial z}{\partial x} = \lim\limits_{\delta x \to 0} \dfrac{f(x+\delta x, y) - f(x, y)}{\delta x}$

In similar manner, we can define $\dfrac{\partial^2 z}{\partial x^2}, \dfrac{\partial^2 z}{\partial x\,\partial y}, \dfrac{\partial^2 z}{\partial y\,\partial x}, \dfrac{\partial^2 z}{\partial y^2}$. Thus, $\dfrac{\partial^2 z}{\partial x^2}$ is nothing but $\dfrac{\partial}{\partial x}\left(\dfrac{\partial z}{\partial x}\right)$; $\dfrac{\partial^2 z}{\partial x\,\partial y}$ is same as $\dfrac{\partial}{\partial x}\left(\dfrac{\partial z}{\partial y}\right)$, $\dfrac{\partial^2 z}{\partial y\,\partial x} = \dfrac{\partial}{\partial y}\left(\dfrac{\partial z}{\partial x}\right)$ and $\dfrac{\partial^2 z}{\partial y^2} = \dfrac{\partial}{\partial y}\left(\dfrac{\partial z}{\partial y}\right)$. In this manner one can define partial derivatives of higher orders.

13. The point $(c, f(c))$ is called a *maximum point* of $y = f(x)$, if (i) $f(c+h) \le f(c)$, and (ii) $f(c-h) \le f(c)$ for small $h \ge 0$. $f(c)$ itself is called a *maximum value* of $f(x)$.
14. A function $f(x)$ is said to be *increasing* (*decreasing*) if $f(x+c) \ge f(x) \ge f(x-c)$ $[f(x+c) \le f(x) \le f(x-c)]$ for all $c \ge 0$.

2.12 QUESTIONS AND EXERCISES

Short-Answer Questions

1. Evaluate $\lim\limits_{x \to 0} \dfrac{\sin 3x}{\sin 5x}$.
2. Differentiate with respect to x; $y = \sin^3 x$.
3. Differentiate with respect to x; $y = \log(\sin x)$.
4. Find derivative $y = \log x$ from first principle (ab initio).
5. Find fifth derivative of $y = x^4$.
6. Find maximum and minimum value of $y = \sin x + \cos x$.
7. Find derivative of $y = \log \sin(2x+3)$.
8. Differentiate $\sin x$ with respect to $\cos x$.

NOTES

Long-Answer Questions

1. Evaluate the following limits:

$$\underset{x\to 0}{\text{Lim}}\ (1+x)\frac{1}{x},\ \underset{x\to 0}{\text{Lim}}\left(\frac{\sin ax}{\sin bx}\right) \text{ where } b \neq 0.$$

2. Show that $\underset{x\to 0}{\text{Lim}}\ \dfrac{x^n - a^n}{x-a} = na^{n-1}$.

3. Evaluate $\underset{x\to 0}{\text{Lim}}\ \dfrac{xe^r - \log(1+x)}{x^2}$.

4. Prove that if $\underset{x\to a}{\text{Lim}}\ f(x)$ exists, it must be unique.

5. Show that the function defined by:

$$\phi(x) = \begin{cases} 0 & \text{for } x \leq 0 \\ \frac{1}{2} - x & \text{for } 0 < x < \frac{1}{2} \\ \frac{1}{2} & \text{for } x = \frac{1}{2} \\ \frac{3}{2} - x & \text{for } \frac{1}{2} < x < 1 \\ 1 & \text{for } x \geq 1 \end{cases}$$

is not continuous at $x = 0,\ \frac{1}{2}$ and 1.

6. Show that $f(x) = x \sin \frac{1}{x},\ x \neq 0$ and $f(0) = 0$ is continuous for all values of x.

7. Evaluate $\underset{x\to 0}{\text{Lim}}\left[\dfrac{\log(1+x)}{x}\right]$.

8. Prove that if $f(x)$, $g(x)$ are continuous functions of x at $x = a$, then $f(x) \pm g(x), f(x)\, g(x)$ and $\dfrac{f(x)}{g(x)}$ [provided $g(a) \neq 0$] are continuous, at $x = a$.

9. Evaluate $\underset{x\to 0}{\text{Lim}}\ \dfrac{\left(\sin^{-1}\frac{4}{3}x\right)}{x}$.

10. Prove that the function $f(x)$ defined as under:

$$\begin{aligned} f(x) &= x \sin\frac{1}{x}, & x \neq 0 \\ &= 0, & x = 0 \end{aligned}$$

is continuous at $x = 0$.

11. Differentiate the following functions with respect to x,

    (*i*) $3x^2 - 6x + 1$, $(2x^2 + 5x - 7)^{5/2}$

    (*ii*) $\dfrac{x-1}{x+1}, \dfrac{ax^2 + hx + g}{hx^2 + bx + f}$

    (*iii*) $e^{\sin x}$, $\log^{\tan x}$, $\log \sin^2 x^3$, $\log a^x$

    (*iv*) $\sqrt{1-x^2}$, $\sin x \cos x$, $\tan x \log x$

    (*v*) $\dfrac{(3x-4)^3(2-x)}{5x^2+1}$

    (*vi*) $\sqrt{1-3\tan^2 x}$

    (*vii*) $\sin\left(\dfrac{\pi}{4} - x\right)$

    (*viii*) $\sqrt{\cos\sqrt{x}}$

12. Differentiate the following with respect to x:

    (*i*) $\sec^{-1} x$, $\cot^{-1} x$, $\text{cosec}^{-1} x$

    (*ii*) $\sec h^{-1} x$, $\text{cosec } h^{-1} x$, $\cot h^{-1} x$

    (*iii*) $(\sin x)^{\log x} + (\log x)^{\cos x}$

    (*iv*) $\sin h^{-1} x$, $\cos h^{-1} x$, $\tan h^{-1} x$

    (*v*) $x = \tan^{-1} t$, $y = t \sin 2t$

13. If $y = (1 - x)^{1/2} (\sin x)\, e^x$, prove that:

$$\frac{dy}{dx} = (1-x)^{1/2} (\sin x)\, e^x \left[1 + \cot x - \frac{1}{2}(1-x)^{-1}\right]^x.$$

14. If $\sqrt{1-x^4} + \sqrt{1-y^4} = k(x^2 - y^2)$, prove that $\dfrac{dy}{dx} = \dfrac{x}{y}\dfrac{\sqrt{1-y^4}}{\sqrt{1-x^4}}$.

    [**Hint.** Put $x^2 = \sin\theta$, $y^2 = \sin\varphi$.]

15. If $x\sqrt{1+y} + y\sqrt{1+x} = 0$, prove that:

$$\frac{dy}{dx} = -\frac{1}{(1+x)^2}$$

16. Differentiate the following with respect to x:

    (*i*) $(x + 1)(x + 2)(x - 1)$ (*ii*) $\dfrac{x^{\sin x}}{\tan x}$

17. Given that $\cos\dfrac{x}{2}\cos\dfrac{x}{2^2}\cos\dfrac{x}{2^3}\ldots ad \inf = \dfrac{\sin x}{2}$,

    prove that $\dfrac{1}{2}\tan\dfrac{x}{2} + \dfrac{1}{2^2}\tan\dfrac{x}{2^2} + \dfrac{1}{2^3}\tan\dfrac{x}{2^3} + \ldots ad \inf = \dfrac{1}{x} - \cot x$.

NOTES

18. Differentiate the following with respect to x, n times:
    (*i*) $\sin^2 x \sin 2x$, $\cos^4 x$, $\cos^2 x \sin^3 x$
    (*ii*) $e^x \log x$, $x^3 \log x$
    (*iii*) $x^{n-1} \log x$
    (*iv*) $\dfrac{x}{(x-a)(x-b)(x-c)}$
    (*v*) $x^3 \cos x$

19. If $y = e^{e^x}$, prove that $e^{-x}\dfrac{d^3y}{dx^3} = \dfrac{d^2y}{dx^2} + \dfrac{2dy}{dx} + y$.

20. If $y = \dfrac{1}{x}\sin x$, show that $\dfrac{d^2y}{dx^2} + \dfrac{2}{x}\dfrac{dy}{dx} + y = 0$.

21. If $y = e^{\sin^2 x}$, prove that $y_2 = \dfrac{y}{2}(1 + 4\cos 2x + \cos 4x)$.

22. Determine the values of constants A and B such that if
    $y = e^x (A \cos x + B \sin x)$, then for all values of x,
    $$\frac{d^2y}{dx^2} - 3\frac{dy}{dx} + 4y = e^x \sin x$$

23. Given that $y = e^{ax} \sin x$, prove that,
    $$\frac{d^2y}{dx^2} - 2a\frac{dy}{dx} + (a^2+1)\,y = 0$$
    Hence or otherwise calculate:
    $y_1(0)$, $y_2(0)$, $y_3(0)$, $y_4(0)$, and $y_5(0)$.

24. If $\cos^{-1}\left(\dfrac{y}{b}\right) = \log\left(\dfrac{x}{n}\right)^n$, show that,
    $$x^2y_{n+2} + (2n+1)xy_{n+1} + 2n^2y_n = 0.$$

25. If $y = e^m \sin^{-1} x$, show that $(1 - x^2)y_2 - xy_1 - m^2y = 0$ and deduce that,
    $(1 - x^2)y_{n+2} - (2n+1)xy_{n+1} - (n^2 + m^2)y_n = 0.$

26. Compute $\dfrac{\partial z}{\partial x}$ and $\dfrac{\partial z}{\partial y}$ for the functions:
    (*i*) $z = (x + y)^2$
    (*ii*) $z = \log (x + y)$
    (*iii*) $z = \dfrac{x}{x^2 + y^2}$
    (*iv*) $z = e^{x^y}$
    (*v*) $z = e^{ax} \sin by$
    (*vi*) $z = \log (x^2 + y^2)$

27. If $z = \tan(y + ax) + (y - ax)^{3/2}$,

    find the value of $\frac{\partial^2 z}{\partial x^2} - a^2 \frac{\partial^2 z}{\partial y^2}$.

NOTES

28. If $\theta = t^n e^{-r^{2/ut}}$, find the value of n which will make:

    $\frac{1}{r^2} \frac{\partial}{\partial r}\left(r^2 \frac{\partial \theta}{\partial r}\right)$ equal to $\frac{\partial \theta}{\partial t}$.

29. If $y = r^m$ where $r^2 = x^2 + y^2 + z^2$, show that

    $\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} + \frac{\partial^2 v}{\partial z^2} = m(m + 1)\, r^{m-2}$.

30. If $u = f(ax^2 + 2hxy + by^2)$, $v = \varphi\,(ax^2 + 2hxy + by^2)$, prove that

    $\frac{\partial}{\partial y}\left(u \frac{\partial}{\partial x}\right) = \frac{\partial}{\partial x}\left(u \frac{\partial v}{\partial y}\right)$.

31. Discuss the maxima and minima of the following functions:

    (*i*) $x^5 - 5x^4 + 5x^3 - 10$

    (*ii*) $x^3 - 3x^2 - 9$

    (*iii*) $x + \frac{1}{x+1}$

    (*iv*) $\frac{(x+3)^2}{x^2+1}$

    (*v*) $x + \sin 2x$ (for $0 \le x \le \pi$)

    (*vi*) $\cos x + \cos 3x$ (for $0 \le x \le \pi$)

32. Show that there is a minimum at $x = 0$ for the function:

    $f(x) = x^4\left(x - \frac{\pi}{2}\right)^2 + \sin^4 x$

2.13 FURTHER READING

Khanna, V.K, S.K. Bhambri, C.B. Gupta and Vijay Gupta. *Quantitative Techniques*. New Delhi: Vikas Publishing House.

Khanna, V.K, S.K. Bhambri and Quazi Zameeruddin. *Business Mathematics*. New Delhi: Vikas Publishing House.

UNIT 3 COORDINATE GEOMETRY

NOTES

Structure

3.0 INTRODUCTION

Descartes, a French mathematician, developed a system of calculating the dimensions of a plane with the help of its points of location or coordinates. A coordinate plane consists of an x-axis and a y-axis; the points of intersection on this plane give the coordinates. The X-axis coordinate is called abscissa, while the Y-axis coordinate is known as ordinate. In this unit, the basic theorems and formulae of coordinate geometry have been introduced. The unit covers problems related to distance formulae, section formulae and the division of a line (internal and external). You will learn to calculate the slope of a non-vertical line. The equations of a line in the slope-intercept form as well as the normal form have been dealt with in this unit. You will also learn how to calculate the distance between a point located on the coordinate plane and a given line. This unit covers the angle between two lines and the area of a triangle formed by lines located on the coordinate plane.

3.1 UNIT OBJECTIVES

After going through this unit, you will be able to:

- Understand the basic concepts and theorems of coordinate geometry
- Understand the application of section and distance formulae

NOTES

- Learn how to determine the slope of a non-vertical line
- Learn how to obtain the equations of a line in slope-intercept form and normal form
- Calculate the distance of a point located in the coordinate plane from a given line
- Learn how to calculate the angle between the two lines
- Learn how to calculate the area of a triangle formed by lines located on a coordinate plane

3.2 COORDINATE GEOMETRY: AN INTRODUCTION

Analytical geometry is also called algebraic or coordinate geometry, is that branch of geometry that is applied to evaluate the properties of plane figures by means of coordinates of points. To do this, we make use of the notations and operations of algebra and analyse the problems discussed in pure geometry systematically to arrive at their solutions.

Relationships between two or more geometrical figures are usually described by more than one variable. Analytical geometry is one of the several methods of studying these relationships.

We shall consider here the concepts of the point and lines in analytical geometry.

3.2.1 The Distance Formula

To find the distance between two points $P_1(x_1, y_1)$ and $P_2(x_2, y_2)$.

Draw $P_1M_1 \perp OX$, $P_2M_2 \perp OX$, $P_1N \perp P_2M_2$ (see Figure 3.1)

$P_1N = OM_2 - OM_1 = x_2 - x_1$

$P_2N = P_2M_2 - P_1M_1 = y_2 - y_1$

In the right angled triangle P_1NP_2, we have, by Pythagoras' theorem,

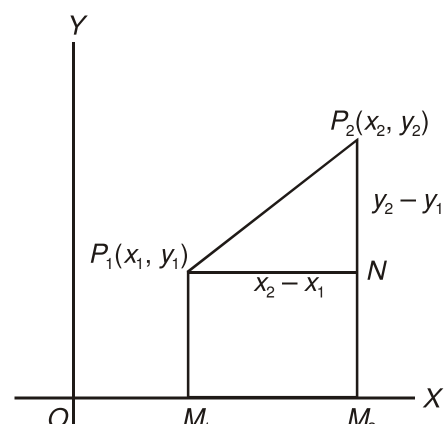


Figure 3.1 *Distance between Two Points P_1, P_2*

$$d^2 = P_1P_2^{\ 2} = P_1N_2 + P_2N_2$$
$$= (x_2 - x_1)^2 + (y_2 - y_1)^2$$

The distance P_1P_2 is given by,

$$d = \sqrt{(x_2 - x_1)^2 \ (y_2 - y_1)^2}$$

Applying this formula, the length of the segment between the points (2, – 1), (– 3, 4) is given by,

$$d = \sqrt{(-3-2)^2 + (4-(-1))^2}$$
$$= \sqrt{25+25} = \sqrt{50} = 5\sqrt{2}.$$

Similarly, the distance between A $(a, 0)$ and B$(0, b)$ is given by (see Figure 3.2):

$$d = \sqrt{(0-a)^2 + (b-0)^2}$$
$$= \sqrt{a^2 + b^2}$$

Which can also be proved by Pythagoras' theorem.

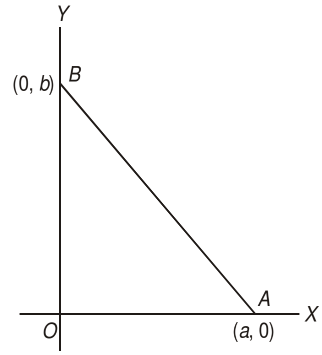


***Figure 3.2** Distance between Two Points A, B*

NOTES

Example 3.1: If the distance of a point $P(x, y)$ from the origin is twice that from (a, b). What is the relation between x, y, a, b?

Solution: Distance of P from $(0, 0)$ is,

$$\sqrt{(x-0)^2 + (y-0)^2} = \sqrt{x^2 + y^2}$$

Distance of P from (a, b) is $\sqrt{(x-a)^2 + (y-b)^2}$

We have, $\sqrt{x^2 + y^2} = 2\sqrt{(x-a)^2 + (y-b)^2}$

$\therefore \quad x^2 + y^2 = 4(x-a)^2 + 4(y-b)^2$

This is the required relation which may be simplified further.

CHECK YOUR PROGRESS

1. Find the lengths of the sides and diagonals of the rectangle formed by the four points $(0, 0)$ (a, b), $(0, b)$ $(a, 0)$.
2. If a circle has the centre $(-5, 1)$ and passes through the point $(-3, -3)$, what is its radius?

3.2.2 Midpoint of a Line Segment

Let $P_1(x_1, y_1)$, $P_2(x_2, y_2)$ be the end points on the line segment P_1P_2 whose midpoint is $P(x, y)$ (see Figure 3.3).

We can easily see that

$$x - x_1 = x_2 - x$$

i.e., $$2x = x_2 + x_1$$

$\therefore$ $$x = \frac{x_1 + x_2}{2}$$

Similarly, $y - y_1 = y_2 - y$

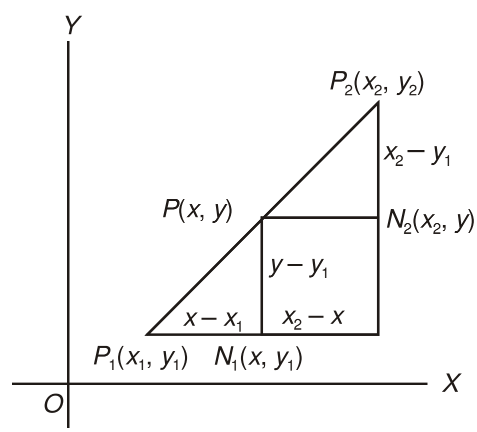


***Figure 3.3** Midpoint of a Line Segment*

NOTES

$$\therefore \qquad y = \frac{y_1 + y_2}{2}$$

The coordinates (x, y) of the midpoint P can be written as:

$$\left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}\right)$$

Applying this formula, the coordinates of the midpoint of the line segment joining (3, 4) and (5, 6) will be:

$$x = \frac{3+5}{2} = 4, \quad y = \frac{4+6}{2} = 5$$

CHECK YOUR PROGRESS

3. Find the coordinates of the centroid G of the triangle whose vertices are (3, 2), (–1, – 4), (– 5, 6). (A centroid is the meeting point of the three medians. It divides each median in the ratio 2: 1).

   Prove that $G = \left(\frac{x_1 + x_2 + x_3}{3}, \frac{y_1 + y_2 + y_3}{3}\right)$, where (x_1, y_1), (x_2, y_2) and (x_3, y_3) are the vertices of the triangle.
4. Given $P(3, 2)$, $Q(5, 8)$, find the coordinates of R on PQ so that
   (*i*) P is the midpoint of QR
   (*ii*) Q is the midpoint of PR
   (*iii*) R is the midpoint of PQ
5. Find the point (x, y) on the y-axis equidistant from (3, 2), (– 5, –2).

3.3 SECTION FORMULA: DIVISION OF LINES

In this section, you will learn to calculate the coordinates of the point that divides a line.

There are two possibilities:

1. P lies on the segment P_1P_2.
2. P lies externally on extended P_1P_2.

3.3.1 Internal Division

When the first of the two possibilities occur, it is called the internal division of a line. Let the line segment P_1P_2 formed by joining $P_1(x_1, y_1)$ and $P_2(x_2, y_2)$ be divided at $P(x, y)$ (see Figure 3.4).

NOTES

Given that,

$$\frac{P_1P}{PP_2} = \frac{k_1}{k_2}$$

Triangles PP_1N_1 and P_2PN_2 are similar,

$$\therefore \quad \frac{P_1P}{PP_2} = \frac{x - x_1}{x_2 - x}$$

$$\therefore \quad \frac{k_1}{k_2} = \frac{x - x_1}{x_2 - x}$$

$$\therefore k_1x_2 - k_1x = k_2x - k_2x_1$$

Or, $\quad k_1x + k_2x = k_1x_2 + k_2x_1$

$$\therefore \quad x = \frac{k_1x_2 + k_2x_1}{k_1 + k_2}$$

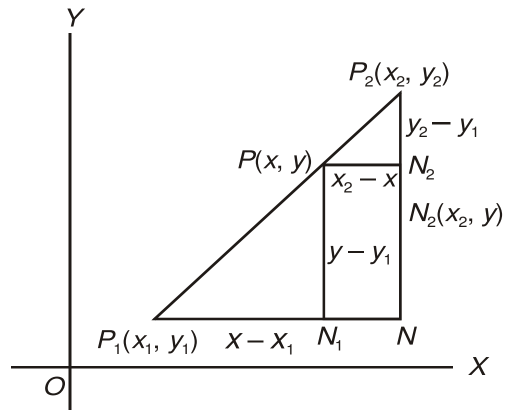


Figure 3.4 *Internal Division*

Similarly, since $\dfrac{P_1P}{PP_2} = \dfrac{y - y_1}{y_2 - y}$ we have,

$$y = \frac{k_1y_2 + k_2y_2}{k_1 + k_2}$$

Applying this foormula, the coordinates of $P(x, y)$ dividing the line segment joining (1, 2) and (3, 4) in the ratio 1 : 3 are given by,

$$x = \frac{k_1x_2 + k_2x_1}{k_1 + k_2} = \frac{1\times 3 + 3\times 1}{1+3} = \frac{3}{2}$$

$$y = \frac{k_1y_2 + k_2y_1}{k_1 + k_2} = \frac{1\times 4 + 3\times 2}{1+3} = \frac{5}{2}$$

3.3.2 External Division

Here, $\dfrac{P_1P}{PP_2} = \dfrac{k_1}{k_2}$

Triangles PP_1N and PP_2N_2 are similar.

$$\therefore \quad \frac{P_1P}{PP_2} = \frac{P_1N}{N_2P_2}$$

$$\therefore \quad \frac{k_1}{k_2} = \frac{x - x_1}{x - x_2}$$

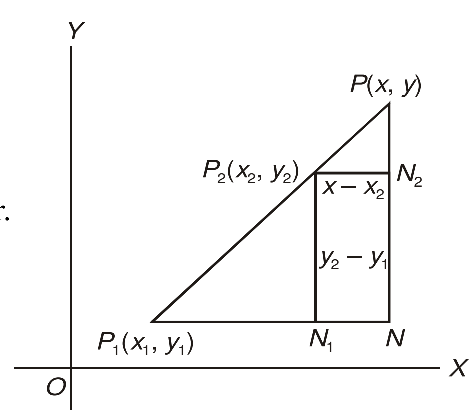


Figure 3.5 *External Division*

NOTES

$$\therefore \quad k_1(x - x_2) = k_2(x - x_1)$$

$$k_1x - k_1x_2 = k_2x - k_2x_1$$

$$\therefore \quad x(k_1 - k_2) = k_1x_2 - k_2x_1$$

Thus, we have $x = \dfrac{k_1x_2 - k_2x_1}{k_1 - k_2}$

Similarly, $\quad y = \dfrac{k_1y_2 - k_2y_1}{k_1 - k_2}$

Applying this formula, the coordinates of $P(x, y)$ dividing the line segment joining (1, 2) and (3, 4) externally in the ratio 5 : 4 are given by,

$$x = \frac{5\times 3 - 4\times 1}{5-4} = 11$$

$$y = \frac{5\times 4 - 4\times 2}{5-4} = 12$$

Hence, we learnt that:

- The coordinates of internal division are $\left(\dfrac{k_1x_2 + k_2x_1}{k_1 + k_2}, \dfrac{k_1y_2 + k_2y_1}{k_1 + k_2}\right)$.
- The coordinates of external division are

$$\left(\frac{k_1x_2 - k_2x_1}{k_1 - k_2}, \frac{k_1y_2 - k_2y_1}{k_1 - k_2}\right) \text{ or } \left(\frac{k_2x_1 - k_1x_2}{k_2 - k_1}, \frac{k_2y_1 - k_1y_2}{k_2 - k_1}\right).$$

Example 3.2: In what ratio does the origin divide line joining (6, 0), (– 3, 0)?

Solution: Let the ratio be k : 1. Origin liss between these two points. Hence, it is an internal division. Then two coordinates of the dividing point, i.e., (0, 0) are

$$x = \frac{k(-3) + 1.6}{k+1} = 0, \ \ y = 0$$

$\therefore -3k + 6 = 0 \quad$ or $\quad k = 2$ the ratio is 2 : 1

Check Your Progress

6. If one end of the diameter of a circle with centre $C(-4, 1)$ is $P(2, 6)$, find the coordinates of the other end of the diameter.
7. Write the coordinates of the point of trisection of the line segment joining (1, 0), (8, 10).
8. Find the ratio of which the Y-axis divides the join of (3, 5), (6, 7).

3.4 EQUATION OF A LINE IN SLOPE-INTERCEPT FORM

NOTES

The graph of a linear function $ax + by + c = 0$, where a, b, c are real numbers, is a straight line.

Every equation of the first degree in x, y represents a straight line. Two points are enough to determine a line.

The equation of a straight line is often written in the following form:

$$y = mx + c$$

In this section, we will derive the equations of a line in slope-intercept form and normal form. Let us first derive the value of the slope of a line.

Let $P_1(x_1, y_1)$, $P_2(x_2, y_2)$ be any two points on a line.

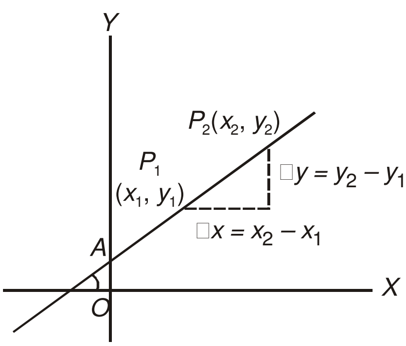


Figure 3.6 *Positive Slope*

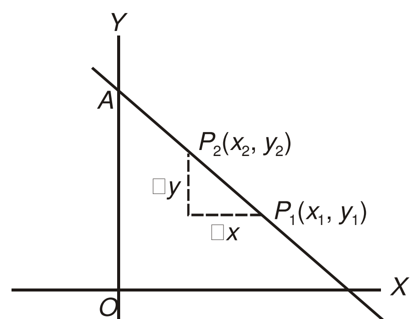


Figure 3.7 *Negative Slope*

Then, the **slope *m* of the line** is defined by:

$$m = \frac{\Delta y}{\Delta x} = \frac{y_2 - y_1}{x_2 - x_1}$$

$$= \frac{\text{Vertical change}}{\text{Horizontal change}}$$

The slope is positive if the line rises to the right (see Figure 3.6). The slope is negative, if the line rises to the left (see Figure 3.7).

Applying this formula, the slope of a line through (4, 7), (2, 1) is

$$m = \frac{y_2 - y_1}{x_2 - x_1} = \frac{1 - 7}{2 - 4} = 3$$

Similarly, if (4, –2) is a point on a line passing through the origin (0, 0) its slope is

$m = \dfrac{-2 - 0}{4 - 0} = -\dfrac{1}{2}$.

Thus, the equation of a line in slop-intercept form is $y = mx + c$.

NOTES

***Note*:** Whenever we take two points on the same line, the slope will be the same quantity m. In each case:

Let (x_1, y_1) be on a line (see Figure 3.8)

$$y = mx + c.$$

Then, $\quad y_1 = mx_1 + c_1.$

If (x_2, y_2) is on the same line, then

$$y_2 = mx_2 + c_2$$

Subtracting

$$y_2 - y_1 = mx_2 - mx_1$$
$$= m(x_2 - x_1)$$

We have the slope $m = \dfrac{y_2 - y_1}{x_2 - x_1}$

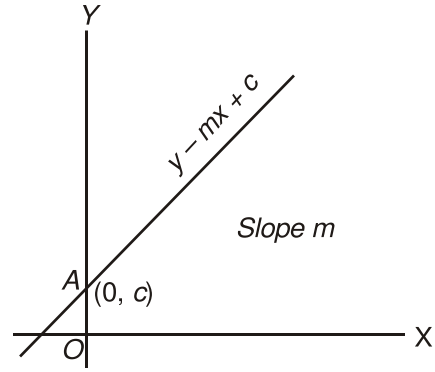


Figure 3.8 *Slope-Intercept Form*

Which is a constant quantity since x_1, y_1, x_2, y_2 are constant quantities.

The slope of the line $y = mx + c$ is the value of the coefficient of x. (Here, it is m)

Also, OA is the intercept made by the line on the y-axis. It is the value of y, where the line cuts the y-axis, i.e., where $x = 0$. (here $OA = c$).

For any equation $ax + by + c = 0$ (see Figure 3.9), we can find the slope by writing it as

$$y = -(a/b)x - c/b$$

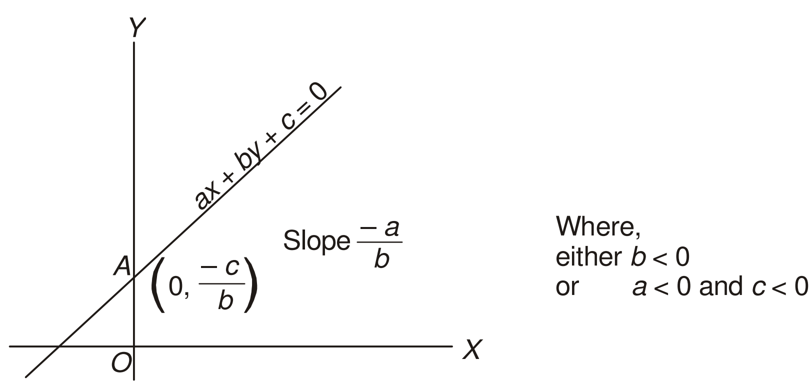


Figure 3.9 *Slope of a Line*

This line is of the form, $y = mx + c$.

$\therefore$ Here, slope $= -\dfrac{a}{b}$ and the intercept $= -\dfrac{c}{b}$.

The Figure 3.9 shows positive slope. Hence, either $b < 0$ and a and c both are positive or $b > 0$ and a and c, both are negative

Applying the formula, if $P_1(4, 7)$, $P_2(2, 1)$ are points on a line, find the slope of the line,

$$m = \frac{y_2 - y_1}{x_2 - x_1} = \frac{1-7}{2-4} = \frac{-6}{-2} = 3.$$

NOTES

Similarly, if $P(4, 2)$ is a point on a line passing through the origin (0, 0), the slope is given by,

$$m = \frac{2-0}{4-0} = \frac{1}{2}.$$

Example 3.3: Find the slope of the line $3y = 9x - 2$.

Solution: The slope can be obtained by writing the line equation as

$$y = \frac{9x}{3} - \frac{2}{3} \quad \therefore \quad m = 3$$

Using the same formula, the slope of the line $x = 2y - 7$ is obtained by writing the line equation as

$$y = \frac{1}{2}x + \frac{7}{2} \quad \therefore \quad m = \frac{1}{2}.$$

Similarly, the slope of the line $\frac{x}{2} - \frac{y}{3} = 1$ is obtained by writing it as

$$y = \frac{3}{2}x - 3 \quad \therefore \quad m = \frac{3}{2}.$$

3.4.1 Variations of Slope-Intercept Form

1. $y = mx + c$

This is the equation of a line with slope m and intercept c on the y-axis.

Let the line cut the y-axis at A (see Figure 3.10 (i)). At A, $x = 0$ so that by substitution,

$$y = m \times 0 + c,$$

$\therefore$ $y = c$

$\therefore$ Coordinates of A are $(0, c)$

At B, $y = 0$ so that $0 = mx + c$

$\therefore$ $x = \frac{c}{m}$

$\therefore$ Coordinates of B are $\left(-\frac{c}{m}, 0\right)$

NOTES

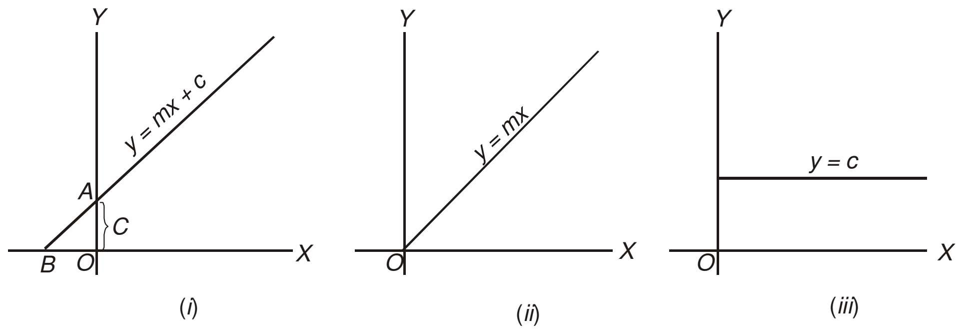


Figure 3.10 *Interceptors Y-axis*

2. If the line passes through the origin, there is no intercept, i.e., $c = 0$.

   $y = mx$ is the equation of a line with slope m, passing through the origin (see Figure 3.10(*ii*)). The slope is $m = \dfrac{y}{x}$.

   If m is positive, the line makes an acute angle with the x-axis.

   If m is negative, the angle is obtuse.

3. Any straight line can be written in the general abstract form of an equation:

$$y = mx + c$$

A line is a particular form of association or relation between two quantities, x and y; x and y are variables whose variations are under consideration.

m is the slope of the line. The slope of a line shows the increase in the value of always y for a unit increase in x. A line has a constant slope.

Also, $\quad c = OA$ intercept on y-axis

$$-\frac{c}{m} = OB \text{ intercept on } x\text{-axis}$$

If $c = 0$, there is no intercept on the y-axis (nor on the x-axis) and the equation reduces to $y = mx$. This is a line passing through the origin (see Figure 3.10 (*ii*)).

4. If $m = 0$, the resulting line $y = c$ is parallel to the x-axis, at a distance c from the y-axis. Its slope is zero. [(see Figure 3.10 (iii)]

m and c are called *parameters* of the equation. They are constant for a given straight line. Different values of m and c will give lines with different slopes and intercepts.

5. The relation $y = mx + c$ is often used in economics as an approximate linear model because, in practice, exact linear relations are not possible.

NOTES

6. Equation of a line with slope m and passing through (x_1, y_1)

   Let (x, y) be any point on a given line.

   The slope of the line is given by

$$m = \frac{y - y_1}{x - x_1}$$

$\therefore \quad y - y_1 = m(x - x_1)$

   This is the equation of the line passing through (x_1, y_1)

7. Equation of a line having intercept a on x-axis and intercept b on y-axis

   The slope of a given line (see Figure 3.11) is,

$$\frac{b-0}{0-a} = -\frac{b}{a}$$

If (x, y) is any point on the line express, we can also find value of the slope as follows:

$$\frac{y-0}{x-a} = \frac{y}{x-a}$$

$$\therefore \quad \frac{y}{x-a} = -\frac{b}{a}$$

$$\frac{y}{b} = \frac{x}{a} + 1$$

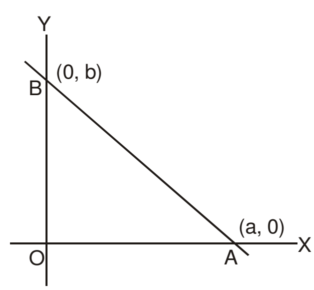


Figure 3.11 *Intercept Form of a Line*

$\therefore \quad \frac{x}{a} + \frac{y}{b} = 1$ is the equation of the line with intercepts a, b on the axes.

8. For two given lines $y = m_1x + c_1$ and $y = m_2x + c_2$ their relationship will be expressed as follows:

(*i*) *Intersecting lines:* The two lines intersect, if there is a value of x which satisfies the two simultaneously.

$$m_1x + c_1 = m_2x + c_2$$

$$\therefore \quad x = \frac{c_2 - c_1}{m_1 - m_2} \; (m_1 \neq m_2)$$

If θ is the angle between two intersecting lines, $\tan\theta = \dfrac{m_2 - m_1}{1 + m_1m_2}$

(*ii*) *Parallel lines:* If $m_1 = m_2$, the lines are parallel. The slopes of parallel lines are equal.

(*iii*) *Identical lines:* If $m_1 = m_2$ and $c_1 = c_2$, the two lines coincide. The slopes and intercepts of identical lines are equal.

NOTES

(*iv*) *Perpendicular lines*: If $m_1 m_2 = -1$, then the lines are perpendicular. The product of the slopes of perpendicular lines is –1.

These relationships can be expressed in following terms for lines,

$$a_1x + b_1y + c_1 = 0 \text{ and } a_2x + b_2y + c_2 = 0.$$

Their slopes are $m_1 = -\frac{a_1}{b_1}, \quad m_2 = -\frac{a_2}{b_2}$.

The lines intersect, if $-\frac{a_1}{b_1} \neq -\frac{a_2}{b_2}$.

The lines are parallel, if $-\frac{a_1}{b_1} = -\frac{a_2}{b_2}$, i.e., if $\frac{a_1}{b_1} = \frac{a_2}{b_2}$.

The lines are identical, if $\frac{a_1}{b_1} = \frac{a_2}{b_2} = \frac{c_1}{c_2}$.

The lines are perpendicular, if $\left(-\frac{a_1}{b_1}\right)\left(-\frac{a_2}{b_2}\right) = -1$.

i.e., $\frac{a_1a_2}{b_1b_2} = -1$ or $a_1a_2 = -b_1b_2$. We can also express this as $a_1a_2 + b_1b_2 = 0$.

3.5 EQUATION OF A LINE IN NORMAL FORM

In this section, we will derive the equation for a given line in normal form.

Let us consider a line AB at a distance p from the origin (see Figure 3.12).

Let $P(x, y)$ be a point on the line.

PM makes an angle θ with the line.

(PM is perpendicular to x-axis, $OQ = p$ is perpendicular to the line and makes angle θ with positive x-axis.)

$$OQ = OA \cos\theta$$
$$= (OM + MA) \cos\theta$$
$$\therefore \quad p = (x + y\tan\theta)\cos\theta$$

$\therefore$ $\boldsymbol{x \cos\theta + y \sin\theta = p}$ is the equation of a line making an angle θ with the x-axis. It is known as the equation of a line in normal form.

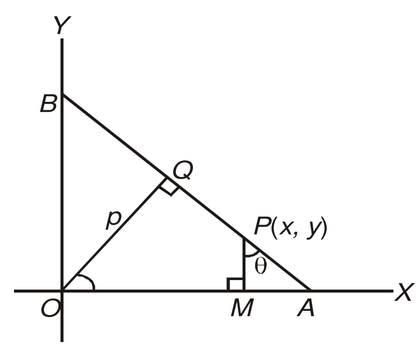


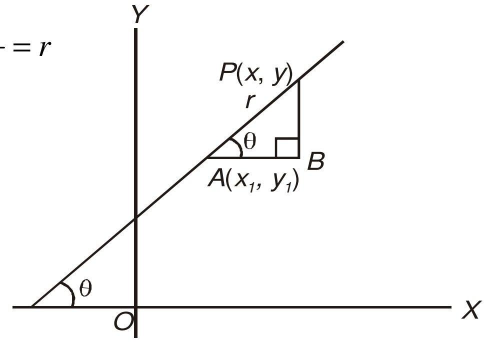
Figure 3.12 *Normal Form of a Line*

For $x = 0$, the y intersect $= p/\sin\theta$.

For $y = 0$, the x intersect $= p/\cos\theta$.

Example 3.4: Find the equation of a line that passes through $A(x_1, y_1)$ and makes an angle θ with x-axis.

Solution: Let $P(x, y)$ be on the line and let $AP = r$ (see Figure below).



$$\cos\theta = \frac{AB}{AP} = \frac{x - x_1}{r} \text{ or } \frac{x - x_1}{\cos\theta} = r$$

$$\sin\theta = \frac{BP}{AP} = \frac{y - y_1}{r}$$

Or, $$\frac{y - y_1}{\sin\theta} = r$$

$\therefore$ The required equation is

$$\frac{x - x_1}{\cos\theta} = \frac{y - y_1}{\sin\theta} = r$$

Thus, $x = x_1 + r\cos\theta$ and $y = y_1 + r\sin\theta$ are the coordinates of any point on the above line at a distance r from A.

NOTES

CHECK YOUR PROGRESS

9. Find the slopes of the lines passing through:
   (*i*) $(-1, 0), (1, 0)$
   (*ii*) $(a, b), (2a, 2b)$
   (*iii*) $(0, 0)\ (2, 5)$
   (*iv*) $(a + k, b + k), (a + m, b + m)$
   (*v*) $(-a, -b), (-b, -a)$

3.5.1 Angle between Two Lines

In this section, we will derive a formula, to determine the value of the angle made by two line.

Given $y = m_1x + c_1$, $y = m_2x + c_2$ making angle α_1, α_2 with the x-axis (see Figure 3.13).

Slopes $\tan\alpha_1 = m_1$, $\tan\alpha_2 = m_2$

$$\alpha_1 - \alpha_2 = \theta \text{ or } \pi - \theta$$

$\therefore \tan(\alpha_1 - \alpha_2) = \tan\theta$

Or, $\tan(\pi - \theta)$ i.e., $-\tan\theta$

$$\therefore \tan\theta = \pm\frac{\tan\alpha_1 - \tan\alpha_2}{1 + \tan\alpha_1 \tan\alpha_2}$$

$$= \pm\frac{m_1 - m_2}{1 + m_1 m_2}$$

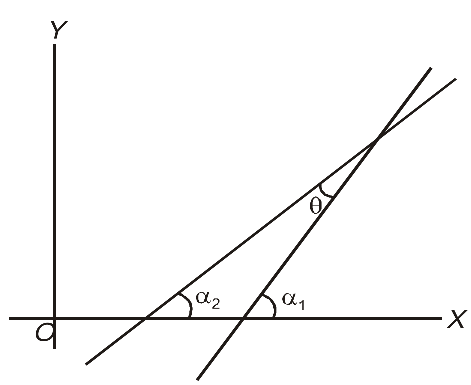


Figure 3.13 *Angle between Two Lines*

***Notes*:**

1. Two lines are parallel, if $\tan\theta = 0$ i.e., $m_1 = m_2$.
2. They are perpendicular, if $\tan\theta = \infty$ i.e., $m_1 m_2 = -1$.
3. The lines $a_1x + b_1y + c_1 = 0$ and $a_2x + b_2y + c_2 = 0$ are parallel, if

NOTES

$m_1 = -a_1/b_1 = -a_2/b_2 = m_2$. The lines are perpendicular, if $\left(-\frac{a_1}{b_1}\right)\left(-\frac{a_2}{b_2}\right) = -1$, i.e., $a_1a_2 + b_1b_2 = 0$.

Example 3.5: What is the equation of a line passing through $(a\cos^3\theta, a\sin^3\theta)$ and perpendicular to the line $x\sec\theta + y\,\mathrm{cosec}\,\theta = a$?

Solution: Slope of the line $= -\dfrac{\sec\theta}{\mathrm{cosec}\,\theta}$

The required line perpendicular to it has slope $= \dfrac{\mathrm{cosec}\,\theta}{\sec\theta}$

$\therefore$ Its equation is $y - a\sin^3\theta = \dfrac{\mathrm{cosec}\,\theta}{\sec\theta}(x - a\cos^3\theta)$

solving and rearranging, we get

$$x\cos\theta - y\sin\theta = a(\cos^4\theta - \sin^4\theta)$$
$$= a(\cos^2\theta - \sin^2\theta)(\cos^2\theta + \sin^2\theta)$$
$$= a\cos 2\theta$$

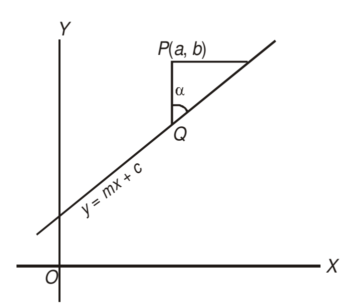
Example 3.6: Find the equation of a line that passes through (a, b) which makes an angle α with $y = mx + c$.

Solution: If this line PQ (see Figure) has slope m_1, then

$$\tan\alpha = \pm\frac{m - m_1}{1 + m\,m_1}$$

$$\therefore \quad m_1 = \frac{m \mp \tan\alpha}{1 \pm m\tan\alpha}$$

There are two lines



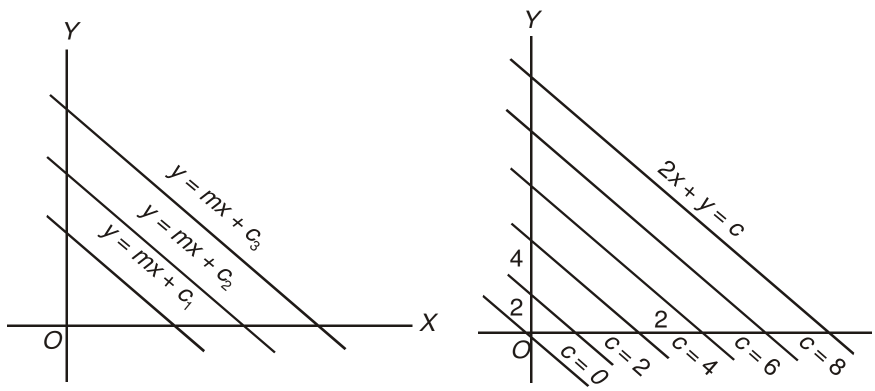
$$y - b = \frac{m - \tan\alpha}{1 + m\tan\alpha}(x - a)$$

$$y - b = \frac{m + \tan\alpha}{1 - m\tan\alpha}(x - a)$$

3.5.2 Families of Parallel Lines

NOTES

A family of parallel lines can be expressed generally by the equation $y = mx + c$, where c can take varying values c_1, c_2,... etc. (see Figure 3.14).



(i) *(ii)*

Figure 3.14 *Family of Parallel Lines*

If some constraints are imposed on the equation, it is possible to find the highest value of the parameter c in a given situation.

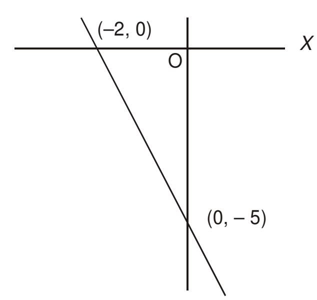
Applying the same concept, if $2x + y = c$, the line giving the highest value of c passes through the upper right vertex of the quadrilateral which is given by

$0 \leq x \leq 4$, $0 \leq y \leq 8$ (see Figure 3.14 (ii)).

This corresponds to c = 8

$2x + y = 8$.

Example 3.7: What is the equation of a line that makes intercepts, – 2 and – 5, on the axes (see Figure)?



Solution: $\dfrac{x}{-2} + \dfrac{y}{-5} = 1$

$\therefore \quad -5x - 2y = 10$

Or, $\quad 5x + 2y + 10 = 0$

NOTES

Check Your Progress

10. What is the equation of a line that meets x-axis at (3, 0) and y-axis at (0, –5)?
11. What is the equation of a line passing through (2, 3) with its intercept on x-axis twice the intercept on y-axis?
12. Find the equation of a line passing through (2, 6) and (5, 3). Find its slope and intercepts.
13. For what values of c will the point (7, c) lie on the line passing through (3, 6) and (– 5, 2)?
14. Two vertices of an equilateral triangle are (– 4, 0), (4, 0). Find the third vertex.

Example 3.8: Consider the slopes of the following lines:

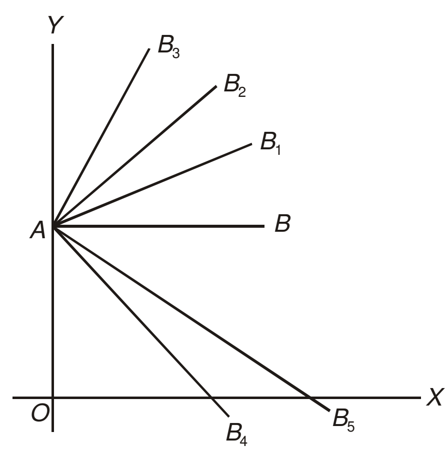
$AB, AB_1, AB_2, AB_3, AY, AO, AB_4, AB_5$

The slope of AB is zero. The slope goes on increasing and the slope of AY is ∞.

The line AO is the same as AY.

The slopes of AB_4, AB_5 are negative. What about the intercepts of these lines?

Solution:



As evident from the Figure above, intercept OA falls on y-axis having no intercept whereas intercepts for other lines fall on x-axis.

Example 3.9: Prove that the figure formed by the points $A(3, 1)$, $B(6, 0)$, $C(4, 4)$ is a right angled triangle.

Solution: Consider the Figure given here

$$\text{Slope of AB} = \frac{0-1}{6-3} = -\frac{1}{3}$$

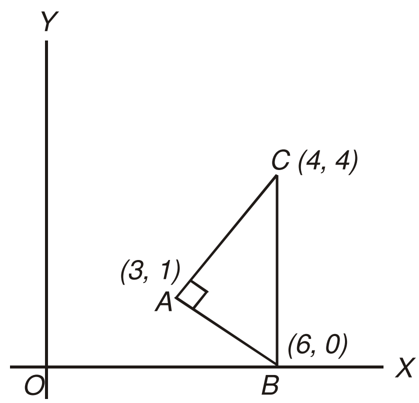
$$\text{Slope of AC} = \frac{4-1}{4-3} = 3$$

$$-\frac{1}{3}\times 3 = -1$$

$\therefore \quad AB \perp AC$

By finding the lengths of the lines. We can prove that,

$$BC^2 = AB^2 + AC^2$$



NOTES

CHECK YOUR PROGRESS

15. Given that $(-3, -2)$, $(7, 4)$, $(1, 14)$ are the vertices of an isosceles right angled triangle, find the length of the perpendicular from the vertex of the right angle to the hypotenuse.
16. (*i*) $(2, 4)$, $(6, 2)$, $(8, 6)$ are three vertices of a square. Find the fourth vertex.

    (*ii*) If $(3, 6)$, $(-5, 2)$, $(7, y)$ are points on the same line find y.
17. (*i*) Find the equation of the line parallel to the line $5x - 7y - 10 = 0$ and passing through $(3, -4)$.

    (*ii*) Find the equations of the medians of a triangle with vertices $(1, 6)$, $(3, -4)$, $(-5, -1)$.
18. Find the equation of the line passing through $(8, 3)$ and through the point of intersection of the lines $5x - 2y + 15 = 0$, $3x + y = 13$.

3.6 DISTANCE OF A POINT FROM A LINE

In this section, you will learn to calculate the distance of a point, located in a coordinate plane, from a given line.

Let us consider a point R with coordinates $(x_1, y_1,)$ Let the line be $ax + by + c = 0$ (see Figure 3.15)

NOTES

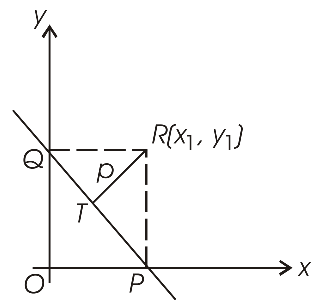


Figure 3.15 *Distance of a Point from a Line*

The equation of line PQ is $ax + by + c = 0$. Draw a perpendicular from R on PQ. RT will give us the distance between point R and line PQ.

Suppose $RT = p$

From equation of the line, we can deduce the coordinates of P and Q, which will be $\left(\frac{-c}{a},0\right)$ and $\left(0,\frac{-c}{b}\right)$ respectively.

By using the distance formula, learnt in previous sections, we have

$$PQ = \sqrt{\left(0+\frac{c}{a}\right)^2+\left(\frac{-c}{b}-0\right)^2}$$

$$= \sqrt{\frac{c^2}{a^2}+\frac{c^2}{b^2}} = \frac{c}{ab}\sqrt{a^2+b^2}$$

Now, join PR and RQ so that ΔPQR is formed.

Area of a triangle with three vertices (x_1, y_1), (x_2, y_2) and (x_3, y_3) is given by

$$\frac{1}{2}\,[x_1(y_2 - y_3) + x_2(y_3 - y_1) + x_3(y_1 - y_2)]$$

Thus, area of ΔPQR $= \left|\frac{1}{2}\left[x_1\left(\frac{-c}{b}-0\right)+0(0-y_1)+\frac{-c}{a}\left(y_1+\frac{c}{b}\right)\right]\right|$

$$= \left|\frac{-1}{2}\left(\frac{x_1c}{b}+\frac{y_1c}{a}+\frac{c^2}{ab}\right)\right|$$

$$= \frac{c}{2ab}\left(ax_1+by_1+c^2\right) \qquad ...(3.1)$$

Now, we know that area of a triangle is also the product of its base and height divded by 2.

Hence, for ΔPQR, its area

$$= \frac{1}{2} \times \text{PQ} \times p. \qquad ...(3.2)$$

NOTES

Equating equation (1) and (2), we get

$$\frac{1}{2}.PQ.p = \frac{c}{2ab}\ (ax_1 + by_1 + c)$$

$$p = \frac{c}{PQ}.\frac{(ax_1 + by_1 + c)}{ab}$$

$$= \frac{c(ax_1 + by_1 + c)}{\frac{c}{ab}\sqrt{a^2 + b^2}.ab}$$

$$= \frac{ax_1 + by_1 + c}{\sqrt{a^2 + b^2}}$$

The length of a line can only be positive.

Hence, $\qquad p = \left|\frac{ax_1 + by_1 + c}{\sqrt{a^2 + b^2}}\right|$

Thus, the distance between point R and line PQ is given by the absolute value of $\frac{ax_1 + by_1 + c}{\sqrt{a^2 + b^2}}$.

3.6.1 Area of a Triangle

Let the coordinates of the vertices (see Figure 3.16) be given by:

A(x_1, y_2), $B(x_2, y_2)$, $C(x_3, y_3)$

It may be remembered that the area of trapezium

$= \frac{1}{2} \times$ Sum of parallel sides

$\times$ Perpendicular distance between them.

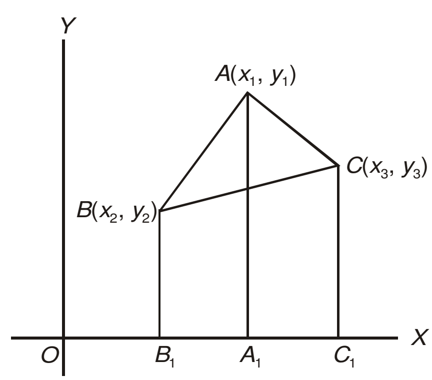


Figure 3.16 *Finding Area of a Triangle*

Thus, $\qquad AA_1B_1B = \frac{1}{2}\ (AA_1 + BB_1)A_1B_1$

$$= \frac{1}{2}\ (y_1 + y_2)\ (x_1 - x_2)$$

NOTES

$$\Delta ABC = \text{Trap. } AA_1B_1B + \text{Trap. } ACC_1A_1 - \text{Trap. } BCC_1B_1$$

$$= \frac{1}{2}(y_1 + y_2)(x_1 - x_2) + \frac{1}{2}(y_1 + y_3)(x_3 - x_1)$$

$$- \frac{1}{2}(y_2 + y_3)(x_3 - x_2)$$

$$= \frac{1}{2}(x_1y_2 - x_2y_1 + x_2y_3 - x_3y_2 + x_3y_1 - x_1y_3)$$

$$= \frac{1}{2}[(x_1(y_2 - y_3) + x_2(y_3 - y_1) + x_3(y_1 - y_2)$$

If three points lie on a straight line, the area of the triangle formed by them is zero, i.e., = 0. Infact, this is the condition for three points to be on a line (collinearity). $= \frac{1}{2}[(x_1(y_2 - y_3) + x_2(y_3 - y_1) + x_3(y_1 - y_2)$

CHECK YOUR PROGRESS

19. Find the area of a triangle with vertices:
    (*i*) (10, 7), (– 2, 3), (0, 0);
    (*ii*) 91, 3) (5, 6), (2, 4)
20. If (0, 4), (– 1, 3), (k, 5) are collinear, find k.

3.7 SUMMARY

In this unit, you have learned that:

- $ax + by + c = 0$ is a linear equation. It is of degree one in x and y. It can be represented geometrically by a straight line.
- If (x_1, y_1), (x_2, y_2) are points on a line, its slope is defined by:

  $$\tan\theta = m = \frac{y_2 - y_1}{x_2 - x_1}$$

  where θ is the smallest positive angle between the lines and the positive x-axis. The rate of change of a linear function $ax + by + c = 0$ is constant and equals the slope of the line.
- There are many forms of the equation of a straight line such as:

  The slope-intercept from $y = mx + c$

  The point-slope form $y - y_1 = m(x - x_1)$

  The two-point form $\dfrac{y - y_1}{y_2 - y_1} = \dfrac{x - x_1}{x_2 - x_1}$

  The intercept form $\dfrac{x}{a} + \dfrac{y}{b} = 1$

The line parallel to y-axis $x = k$

The line parallel to x-axis $y = k$

NOTES

- In the general form $ax + by + c = 0$, the slope $m = -\frac{a}{b}$ and the intercepts are $-\frac{c}{a}$ on x-axis and $-\frac{c}{b}$ on y-axis.

  If $a = 0$, the line is parallel to x-axis;

  if $b = 0$, the line is parallel to y-axis.
- If two lines with slopes m_1, m_2 are parallel, then $m_1 = m_2$.

  If they are perpendicular $m_1 m_2 = -1$
- If θ is the angle between two perpendicular lines, then $\tan\theta = \left|\frac{m_2 - m_1}{1 + m_1 m_2}\right|$
- The coordinates of the point of intersection for two intersecting lines, obtained by solving the equations, satisfy the equations of both the lines.
- Three given lines have a common point of intersection if the coordinates of the intersection point on any two lines satisfy the equation of the third line.
- The normal form of the equation of a line is given by:

  $$x\cos\theta + y\sin\theta = p$$
- The perpendicular distance of a given points (x_1, y_1) from a given line $ax + by + c = 0$ is given by:

  $$p = \left|\frac{ax_1 + by_1 + c}{\sqrt{a^2 + b^2}}\right|$$

  The perpendicular from the origin to this line is $p = \left|\frac{c}{\sqrt{a^2 + b^2}}\right|$
- Distance between parallel lines is the perpendicular distance of any point on one of the lines from the other line.
- Any line through the intersection of two given lines,

  $a_1x + b_1y + c_1 = 0$ and $a_2x + b_2y + c_2 = 0$ is expressed by:

  $a_1x + b_1y + c_1 + k(a_2x + b_2y + c_2) = 0$

  Which represents the family of all lines passing through the point of intersection of the given lines.

3.8 KEY TERMS

NOTES

- **Coordinate geometry:** It is the branch of mathematics that deals with the evaluation of properties of plane figures by means of their location, in terms of their coordinates, on a coordinate plane.
- **External division of a line:** When the point dividing a particular line falls on the line only when it is extended, it is said to be an external division of that line.
- **Intercept:** It is the distance between the origin and the points at which a particular line intersects the two axes.

3.9 ANSWERS TO 'CHECK YOUR PROGRESS'

1. $a, b, a, b;\ \sqrt{a^2+b^2}, \sqrt{a^2+b^2}$
2. $\sqrt{17}$
3. (– 1, 4/3)
4. (*i*) (1, – 4)
   (*ii*) (7, 14)
   (*iii*) (4, 5)
5. (0, – 2)
6. (– 10 – 4)
7. $\left(\frac{13}{3}, \frac{14}{3}\right), \left(\frac{17}{3}, \frac{22}{3}\right)$
8. External division – 1: 2
9. (*i*) 0
   (*ii*) *b*/*a*
   (*iii*) 5/2
   (*iv*) 1
   (*v*) –1
10. $\frac{x}{3}+\frac{y}{-5}=1$ or $\frac{x}{3}-\frac{y}{5}=1$ or $5x - 3y = 15$
11. $x + 2y - 8 = 0$
12. $x + y = 8$; – 1; 8, 8
13. 8

14. $(0, \pm 4\sqrt{3})$

15. $2\sqrt{17}$

16. (i) (4, 8)

    (ii) The slopes of the lines joining any two of these three points will have the same value of that $y = 8$

17. (i) $17x - 14y + 7 = 10$,

    (ii) $13x + 10y + 1 = 0$, $2x - 7y + 3 = 0$

18. $x + y = 11$

19. (*i*) 22

    (*ii*) $\dfrac{1}{2}$

20. 1

NOTES

3.10 QUESTIONS AND EXERCISES

Short-Answer Questions

1. Find the third vertex if two vertices of an equilateral triangle are given to be (0, 0), (– 4, 3).
2. Find the midpoint of $(-a, 0)$, $(4, 0)$.
3. Prove that $A(0, 0)$, $B(10, -4)$ $C(2, 5)$ form a right angled triangle.
4. The intercepts of parallel lines are different but the slopes are the same. Can lines with negative and positive intercepts on the same axis be parallel?
5. Show that $(0, 0)$, $(a, 0)$, $a + (\sqrt{a^2 - b^2}, b)$, $(\sqrt{a^2 - b^2}, b)$ are vertices of a rhombus $(a > b)$. Show that the diagonals are perpendicular.
6. Show that $y = 7x + 2$, $y = \dfrac{1}{7}x + 4$ are perpendicular lines.

   Hint. $m_1 m_2 = 7 \times \dfrac{1}{2} = -1$
7. Show that $(3, -2)$, $(4, 3)$, $(-1, 1)$, $(-2, -4)$ are the vertices of a parallelogram. Find the lengths of the diagonals.

   (**Hint:** Find the slopes of opposite pairs of sides and show they are parallel and equal.)
8. Show that the lines $3x + by + 5 = 0$, $cx - 3y - 2 = 0$ are perpendicular, if $3c - 3b = 0$ or $b = c$.

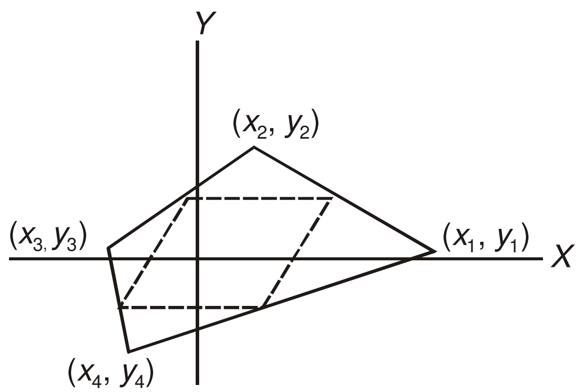
9. Show that if vertex C in a triangle is the origin (0, 0), the area of the triangle is $\frac{1}{2}(x_1y_2 - x_2y_1)$.

NOTES

10. Show that the area of a quadrilateral $A(x_1, y_1)$, $B(x_2, y_2)$ $C(x_3, y_3)$, $D(x_4, y_4)$ is $\frac{1}{2}(x_1y_2 - x_2y_1 + x_2y_3 - x_3y_2 + x_3y_4 - x_4y_3 - x_4y_1 - x_1y_4)$ (sum of two triangles).
11. Show that points (0, 2), (1, 5), (– 1, – 1) are collinear.

Long-Answer Questions

1. Given $A(-1, -1)$, $B(3, -3)$, $C(2, 3)$, find the lengths of the sides and the medians of ΔABC. Also, find the coordinates of the midpoints of the medians.
2. The segments joining the midpoints of the sides of a quadrilateral form a parallelogram. Find the lengths of sides joining the midpoints. (See the following figure).



3. Prove that the diagonals of a square are equal.
4. Prove that the diagonals of a parallelogram bisect each other.
5. Show that the distance from (1, 1) to $\left(\frac{2t^2}{1+t^2}, \frac{(1-t)^2}{1+t^2}\right)$ is same for all values of t. (Show that the answer does not contain t.)
6. Prove that the slope of each of the following lines is zero.

   (*i*) y = 0, (*ii*) y = k, (*iii*) y = – k.

   (Note that $y = 0$ is the equation of y-axis).
7. Prove that the slope of lines (*i*) $x = 0$, (*ii*) $x = k$, (*iii*) $x = -k$ is infinite.

   (Note that $x = 0$ is the equation of y-axis)

NOTES

8. Find the slopes of the following lines:

(*i*) $\frac{x+2}{3} - 4y = \frac{1}{2}$ (*ii*) $x - \sqrt{3y} = 6$

(*iii*) $\sqrt{3x} + y = 12$ (*iv*) $y - 12x = 0$

(*v*) $x = y$ (*vi*) $y\sqrt{2} = x\sqrt{6}$

Are there any parallel or perpendicular lines among these?

Draw the graphs and check.

9. Prove that the following pairs of lines are parallel:

(*i*) $y = 3x + 7$, $y = 3x - 5$ (*ii*) $y = 7x$, $y = 7x + \frac{1}{2}$ (*iii*) $y = x$, $y = x + a$

(*iv*) $ax + by + c = 0$, $bx + k = \frac{b^2}{a}y$ (*v*) $x = k$, $x = 0$ (*vi*) $y = 5$, $y = -5$.

10. Prove that the following pairs of lines are perpendicular:

(*i*) $y = 3x + 7$, $x = 5 - 3y$ (*ii*) $y = 2x$, $y = \frac{3}{4} - \frac{x}{2}$ (*iii*) $\frac{x}{a} + \frac{y}{b} = 1$, $ax - by = 10$

(*iv*) $ax + by + c = 0$, $ay = bx$ (*v*) $y = x$, $x = -y$.

11. (*i*) Prove that the lines joining the midpoints of opposite sides of a quadrilateral bisect each other.

(*ii*) Prove that the diagonals of a rhombus are perpendicular.

12. Show that $\left(x_1, \frac{c+ax_1}{b}\right), \left(x_2, \frac{c+ax_2}{b}\right)$ are points on the line $ax + by = 0$.

(Substitute each point in the equation and see if it is satisfied).

3.11 FURTHER READING

Khanna, V.K, S.K. Bhambri, C.B. Gupta and Vijay Gupta. *Quantitative Techniques.* New Delhi: Vikas Publishing House.

Khanna, V.K, S.K. Bhambri and Quazi Zameeruddin. *Business Mathematics.* New Delhi: Vikas Publishing House.

UNIT 4 QUADRATIC EQUATIONS

NOTES

Structure

4.0 INTRODUCTION

In mathematics, a quadratic equation is a polynomial equation of the second degree. The general form is

$$Ax^2 + bx + c = 0$$

where x represents a variable, and a, b, and c, constants, with $a \neq 0$. (If $a = 0$, the equation is a linear equation.)The three constants—a, b and c— are called the quadratic coefficient, the linear coefficient and the constant term or free term, respectively. The term 'quadratic' is derived from *quadratus*, which in Latin means 'square'. Quadratic equations can be solved by using factorization method, perfect square method and discriminate method. This unit describes all these methods of solving quadratic equations.

A quadratic equation with real or complex coefficients has two solutions, called roots. These two solutions may or may not be distinct, and they may or may not be real. This unit will introduce you to the realation of roots of a quadratic equation.

4.1 UNIT OBJECTIVES

After going through this unit, you will be able to:

- Understand the basics of quadratic equations
- Solve quadratic equations by using factorization and perfect square methods

- Understand the nature of roots
- Comprehend the relation of the roots

NOTES

4.2 QUADRATIC EQUATION: BASICS

An equation of degree 2 is called a *quadratic equation*.

Note: In this section, we shall mainly deal with quadratic equations having rational numbers as coefficients.

There are two types of quadratic equations: (1) Pure and (2) Affected.

A quadratic equation is called pure if it does not contain single power of x. In other words, in a pure quadratic equation, coefficient of x must be zero. Thus a pure quadratic equation is of the type $ax^2 + b = 0$ with $a \neq 0$.

A quadratic equation which is not pure is called an *affected quadratic equation*.

Thus, the most general form of an affected quadratic equation is $ax^2 + bx + c = 0$, with $ab \neq 0$. (Recall that $ab \neq 0 \Leftrightarrow a \neq 0$ and $b \neq 0$).

Root. A complex number α is called a root of $ax^2 + bx + c$ if $a\alpha^2 + b\alpha + c = 0$.

4.2.1 Method of Solving Pure Quadratic Equations

Let $ax^2 + b = 0$ be a pure quadratic equation. This implies

$$ax^2 = -b \Rightarrow x^2 = -\frac{b}{a} \Rightarrow x = \pm\sqrt{\frac{-b}{a}}$$

It is clear that the roots of $ax^2 + b$ are real if and only if a and b are of opposite signs.

Example 4.1: Solve $9x^2 - 4 = 0$.

Solution: Clearly, $9x^2 = 4 \Rightarrow x^2 = \frac{4}{9} \quad \Rightarrow \quad x = \pm\frac{2}{3}$.

4.3 SOLVING QUADRATIC EQUATIONS

Note: Since a pure quadratic equation is a particular case of $ax^2 + bx + c = 0$. All these methods are applicable to pure equations also. All that we have to do is to just put $b = 0$ to get the solution of a pure equation.

4.3.1 Method of Factorization

If the expression $ax^2 + bx + c$ can be factored into linear factors then each of the factors, put to zero, provides us with a root of the given quadratic equation.

Thus, if $ax^2 + bx + c = a(x - \alpha)(x - \beta)$, then the roots of $ax^2 + bx + c = 0$ are α and β.

Example 4.2: Solve $x^2 - 5x + 6 = 0$.

Solution: Clearly, $x^2 - 5x + 6 = 0$

$$\Rightarrow \quad (x-2)(x-3) = 0$$

$$\Rightarrow x - 2 = 0 \quad \text{or} \quad x - 3 = 0$$

$$\Rightarrow \quad x = 2 \quad \text{or} \quad x = 3$$

Hence, roots of given equation are 2 and 3.

NOTES

4.3.2 Method of Perfect Square

This method is made clear by the following steps. Let $ax^2 + bx + c = 0$ be the given equation.

Step 1. Divide both sides of the equation by a to obtain

$$x^2 + \frac{b}{a}x + \frac{c}{a} = 0$$

(since $a \neq 0$, we are justified in division by a)

Step 2. Transpose the constant term (i.e., the term independent of x) on RHS to get,

$$x^2 + \frac{b}{a}x = -\frac{c}{a}$$

Step 3. Add $\frac{b^2}{4a^2}$ to both the sides.

Thus, we have $\quad x^2 + \frac{b}{a}x + \frac{b^2}{4a^2} = \frac{b^2}{4a^2} - \frac{c}{a}$

Or $\quad \left(x + \frac{b}{2a}\right)^2 = \frac{b^2 - 4ac}{4a^2}$

This is a pure equation in the variable $x + \frac{b}{2a}$.

So, the solution is $\quad x + \frac{b}{2a} = \frac{\pm\sqrt{b^2 - 4ac}}{2a}$

Or, $\quad x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$

***Note*:** This method is useful particularly when $ax^2 + bx + c$ cannot be factored into linear factor easily.

Example 4.3: Solve $2x^2 + 3x - 1 = 0$.

Solution: In this case $a = 2$, $b = 3$, $c = -1$

Hence, roots are $x = \frac{-3 \pm \sqrt{3^2 - 4(2)(-1)}}{2.2} = \frac{-3 \pm \sqrt{17}}{4}$

4.4 DISCRIMINANT METHOD AND NATURE OF ROOTS

NOTES

The roots of $ax^2 + bx + c = 0$ are given by $\dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$. The expression inside the radical sign, i.e., $b^2 - 4ac \ \forall\ a, b, c \in R$ is called *discriminant.*

Case 1. $b^2 - 4ac > 0$, i.e., $b^2 > 4ac$.

In this case $\sqrt{b^2 - 4ac}$ is a real number. Hence the two roots of the given equation are unequal and real.

Case 2. $b^2 - 4ac = 0$, i.e., $b^2 = 4ac$.

In this case both the roots are real and equal (each equal to $-b/2a$).

Case 3. $b^2 - 4ac < 0$, i.e., $b^2 < 4ac$.

In this case $\sqrt{b^2 - 4ac}$ is an imaginary number and so both the roots are complex and unequal.

Example 4.4: *Solve* $\dfrac{x+3}{x+2} + \dfrac{x-3}{x-2} = \dfrac{2x-3}{x-1}$.

Solution: Given equation is equivalent to

$$\frac{(x+2)+1}{x+2} + \frac{(x-2)-1}{x-2} = \frac{2(x-1)-1}{x-1}$$

$$\Rightarrow \quad 1 + \frac{1}{x+2} + 1 - \frac{1}{x-2} = 2 - \frac{1}{x-1}$$

$$\Rightarrow \quad \frac{x-2-x-2}{x^2-4} = -\frac{1}{x-1}$$

$$\Rightarrow \quad \frac{-4}{x^2-4} = -\frac{1}{x-1}$$

$$\Rightarrow \quad 4x - 4 = x^2 - 4$$

$$\Rightarrow \quad x^2 - 4x = 0 \Rightarrow x(x-4) = 0 \quad \Rightarrow \quad x = 0 \text{ or } 4.$$

Hence, the roots of the given equation are 0 and 4.

Example 4.5: Solve $x^4 - 13x^2 + 36 = 0$.

Solution: This is not a quadratic equation in x, but on putting $x = t^2$, we get a quadratic in t, namely $t^2 - 13t + 36 = 0$.

Roots of this equation are given by $(t-4)(t-9) = 0$.

Thus, $t = 4$ or $t = 9$. In other words, $x^2 = 4$ or $x^2 = 9$. Hence $x = \pm 2$ or ± 3.

Consequently, roots of given equation are $\pm 2, \pm 3$.

Example 4.6: Solve $(x + 1)(x + 3)(x + 4)(x + 6) = 72$.

Solution: Rearrange the factors on the LHS so as to have the sum of constants in first two factors same as in the case of other two factors.

NOTES

Since $1 + 6 = 3 + 4$, we get $(x + 1)(x + 6)(x + 3)(x + 4) = 72$

Or, $(x^2 + 7x + 6)(x^2 + 7x + 12) = 72$

Now put, $x^2 + 7x = t$ to obtain

$$(t + 6)(t + 12) = 72$$

$$\Rightarrow \quad t^2 + 18t + 72 = 72$$

$$\Rightarrow \quad t(t + 18) = 0 \quad \Rightarrow \quad t = 0 \quad \text{or} \quad t = -18$$

Hence, $x^2 + 7x = 0$ or $x^2 + 7x + 18 = 0$

First, quadratic has 0 and – 7 as its roots and the second quadratic has roots given by

$$\frac{-7 \pm \sqrt{49 - 72}}{2}, \text{ i.e., } \frac{-7 \pm \sqrt{-23}}{2}$$

Example 4.7: Solve $\sqrt{5x^2 - 6x + 8} - \sqrt{5x^2 - 6x - 7} = 1$

Solution: Consider $(5x^2 - 6x + 8) - (5x^2 - 6x - 7) = 15$

Divide this equation by the given equation.

We get,

$$\sqrt{5x^2 - 6x + 8} + \sqrt{5x^2 - 6x - 7} = 15$$

Adding this equation to the given equations, we obtain,

$$2\sqrt{5x^2 - 6x + 8} = 16$$

$$\Rightarrow \quad 5x^2 - 6x + 8 = 64$$

$$\Rightarrow \quad 5x^2 - 6x - 56 = 0$$

$$\Rightarrow \quad x = \frac{6 \pm \sqrt{36 + 1120}}{10} = \frac{6 \pm \sqrt{1156}}{10}$$

$$\Rightarrow \quad x = \frac{6 \pm 34}{10} \quad \Rightarrow \quad x = 4 \quad \text{or} \quad -2\frac{4}{5}.$$

Example 4.8: Solve $x^4 - 5x^3 + 15x + 9 = 0$.

Solution: Note that in this equation

$$x^4 - 5x(x^2 - 3) + 9 = 0$$

$$(x^4 - 6x^2 + 9) - 5x(x^2 - 3) + 6x^2 = 0$$

$$(x^2 - 3)^2 - 5x(x^2 - 3) + 6x^2$$

Put $x^2 - 3 = t$.

Thus, the given equation is reduced to $t^2 - 5xt + 6x^2 = 0$

This has the roots $t = 2x$ and $t = 3x$.

In other words, we have two quadratic equations,

$$x^2 - 3 = 2x \quad \text{and} \quad x^2 - 3 = 3x$$

The roots of former equation are – 1 and 3 and those of the latter are $\frac{3 \pm \sqrt{21}}{2}$.

NOTES

Example 4.9: Solve $5^x + 5^{2-x} = 26.$

Solution: Multiplying the given equation by 5^x we obtain

$$5^{2x} + 25 = 26.5^x$$

Or, $5^{2x} - 26.5^x + 25 = 0$

Put $5^x = t$ to obtain the quadratic equation $t^2 - 26t + 25 = 0.$

The roots of this equation are $t = 1$ or $t = 25.$

Then, $5^x = 1 = 5^0 \Rightarrow x = 0$

Or, $5^x = 25 = 5^2 \Rightarrow x = 2$

Hence, $x = 0$ or $2.$

Example 4.10: Solve $3x - 4 = \sqrt{2x^2 - 2x + 2}$.

Solution: Squaring both sides to eliminate the radical sign, we get

$$9x^2 - 24x + 16 = 2x^2 - 3x + 2$$

Or $7x^2 - 21x + 14 = 0$

Or, $x^2 - 3x + 2 = 0$

$\Rightarrow$ $x = 1$ or 2

Hence, the roots of given equation are 1 and 2.

Example 4.11: Solve $x^4 + x^3 - 4x^2 + x + 1 = 0.$

Solution: In equations of such type if the terms are arranged according to descending powers of x, the coefficients of terms equidistant from first and last term are equal or differ in sign. Equations of this type are called reciprocal equations.

We collect equidistant terms together.

Thus, the given equation is equivalent to

$$(x^4 + 1) + (x^3 + x) - 4x^2 = 0$$

Divide by x^2 to obtain

$$\left(x^2 + \frac{1}{x^2}\right) + \left(x + \frac{1}{x}\right) - 4 = 0$$

Now, put $x + \frac{1}{x} = t$. Then $x^2 + \frac{1}{x^2} = t^2 - 2$

We get, $t^2 - 2 + t - 4 = 0$

Or, $t^2 + t - 6 = 0 \Rightarrow t = -3$ or 2

In other words, $x + \frac{1}{x} = -3$ or 2

i.e., $x^2 + 3x + 1 = 0$ or $x^2 - 2x + 1 = 0$

$\Rightarrow$ $x = \frac{-3 \pm \sqrt{5}}{2}$ or $x = 1, 1$

Hence, the roots of given equation are

$$1, 1, \frac{-3 \pm \sqrt{5}}{2}.$$

Example 4.12: Solve the equation

$$x^2 - 6x + 9 = 4\sqrt{x^2 - 6x + 6}$$

Solution: Putting $x^2 - 6x + 6 = t$ in the given equation, we get

NOTES

$$t + 3 = 4\sqrt{t}$$

Or, $t^2 + 6t + 9 = 16t$

Or, $t^2 - 10t + 9 = 0$

$\Rightarrow$ $(t - 1)(t - 9) = 0$

$\Rightarrow$ $t = 1$ or $t = 9$

$\Rightarrow$ $x^2 - 6x + 6 = 1$ or $x^2 - 6x + 6 = 9$

$\Rightarrow$ $x^2 - 6x + 5 = 0$ or $x^2 - 6x - 3 = 0$

$\Rightarrow$ $(x - 1)(x - 5) = 0$ or $x = \dfrac{6 \pm \sqrt{36 - 4(-3)}}{2}$

$\Rightarrow$ $x = 1, 5$ or $x = \dfrac{6 \pm 4\sqrt{3}}{2}$

$\Rightarrow$ $x = 1, 5$ or $3 \pm 2\sqrt{3}$.

Example 4.13: Solve $\sqrt{\dfrac{x}{1-x}} + \sqrt{\dfrac{1-x}{x}} = 2\dfrac{1}{6}$.

Solution: Putting $\dfrac{x}{1-x} = t^2$, we get

$$t + \frac{1}{t} = \frac{13}{6} \Rightarrow 6t^2 + 6 = 13t$$

$$\Rightarrow 6t^2 - 13t + 6 = 0$$

$$\Rightarrow 6t^2 - 4t - 9t + 6 = 0$$

$$\Rightarrow (2t - 3)(3t - 2) = 0$$

$$\Rightarrow t = \frac{3}{2} \text{ or } \frac{2}{3}$$

Now, $t = \dfrac{3}{2}$ $\Rightarrow \dfrac{x}{1-x} = \dfrac{9}{4}$ $\Rightarrow 4x = 9 - 9x$

$$\Rightarrow 13x = 9 \Rightarrow x = \frac{9}{13}$$

When, $t = \dfrac{2}{3}$ $\Rightarrow \dfrac{x}{1-x} = \dfrac{4}{9} \Rightarrow 9x = 4 - 4x$

$$\Rightarrow 13x = 4 \Rightarrow x = \frac{4}{13}$$

So, $x = \dfrac{4}{13}$ or $\dfrac{9}{13}$.

NOTES

Example 4.14: Find the value of $\sqrt{6+\sqrt{6}+\sqrt{6}+\ldots\infty}$.

Solution: Let $x = \sqrt{6+\sqrt{6}+\sqrt{6}+\ldots\infty} = \sqrt{6+x} \Rightarrow x^2 = 6 + x$

$$\Rightarrow x^2 - x - 6 = 0$$

$$\Rightarrow (x-3)(x+2) = 0$$

$$\Rightarrow x = 3 \quad \text{or} \quad -2.$$

Example 4.15: Solve $\frac{x-p}{q} + \frac{x-q}{p} = \frac{q}{x-p} + \frac{p}{x-q}$.

Solution: Given equation can be rewritten as

$$\frac{x-p}{q} - \frac{q}{x-p} = \frac{p}{x-q} - \frac{x-q}{p} \Rightarrow \frac{(x-p)^2 - q^2}{q(x-p)} = \frac{p^2-(x-q)^2}{p(x-q)}$$

$$\Rightarrow \frac{(x-p-q)(x-p+q)}{q(x-p)}$$

$$= \frac{(p+x-q)(p-x+q)}{p(x-q)}$$

Either $x - p - q = 0$, i.e., $x = p + q$

Or, we get

$$\frac{x-p+q}{q(x-p)} = \frac{-(p+x-q)}{p(x-q)}$$

Simplifying, we get $(p+q)x^2 - (p^2+q^2)x = 0$

$$\Rightarrow \qquad x = 0 \text{ or } x = \frac{p^2+q^2}{p+q}$$

Hence, $x = 0$ or $\frac{p^2+q^2}{p+q}$ or $p + q$.

Example 4.16: Solve $x + \sqrt{x} = \frac{6}{25}$.

Solution: Putting $\sqrt{x} = t$, we get

$$t^2 + t = \frac{6}{25} \quad \Rightarrow \quad 25t^2 + 25t - 6 = 0$$

$$\Rightarrow t = \frac{-25 \pm \sqrt{625-4(-6)(25)}}{50}$$

$$= \frac{-25 \pm \sqrt{625+600}}{50}$$

$$= \frac{-25 \pm \sqrt{1225}}{50} = \frac{-25 \pm 35}{50}$$

$$= \frac{10}{50} \text{ or } \frac{-60}{50}$$

$$= \frac{1}{5} \text{ or } \frac{-6}{5}$$

Then, $x = t^2 = \frac{1}{25}$ or $\frac{36}{25}$.

NOTES

Example 4.17: Solve $x^{2/3} + x^{1/3} - 2 = 0$.

Solution: Put $x^{1/3} = t$, to obtain

$$t^2 + t - 2 = 0 \quad \Rightarrow \quad (t + 2)(t - 1) = 0$$

$$\Rightarrow \quad t = 1 \quad \text{or} \quad -2$$

In case $t = 1$, we get $x^{1/3} = 1 \Rightarrow x = 1$

In case $t = -2$, we get $x^{1/3} = -2 \Rightarrow x = -8$

Hence, $x = 1$ or -8.

Example 4.18: Solve $x^2 + x + 10\sqrt{x^2+3x+16} = 2(20 - x)$.

Solution: Given equation can be written as

$$x^2 + 3x - 40 + 10\sqrt{x^2 + 3x + 16} = 0$$

Put, $\sqrt{x^2 + 3x + 16} = t$

Then, $x^2 + 3x = t^2 - 16$

So, the given equation simplifies to

$$t^2 - 16 - 40 + 10t = 0$$

Or, $t^2 + 10t - 56 = 0$

$\Rightarrow \quad (t + 14)(t - 4) = 0$

$\Rightarrow \quad t = 4 \quad \text{or} \quad -14$

Now, $t = 4 \Rightarrow x^2 + 3x + 16 = 16$

$$\Rightarrow x^2 + 3x = 0 \quad \Rightarrow \quad x = 0 \quad \text{or} \quad -3$$

While, $t = -14 \quad \Rightarrow x^2 + 3x + 16 = 196$

$$\Rightarrow x^2 + 3x - 180 = 0$$

$$\Rightarrow x = \frac{-3 \pm \sqrt{9 + 720}}{2}$$

$$= \frac{-3 \pm \sqrt{729}}{2}$$

$$= \frac{-3 \pm 27}{2} = 12 \ \text{or} \ -15$$

Hence, $x \Rightarrow 0, -3, 12, -15$.

Example 4.19: Solve $3x^2 - 18 + \sqrt{3x^2 - 4x - 6} = 4x$.

Solution: Putting $\sqrt{3x^2 - 4x + 6} = t$, we get

$$3x^2 - 4x = t^2 + 6$$

So, the given equation is reduced to

$$t^2 + 6 - 18 + t = 0 \Rightarrow t^2 + t - 12 = 0$$

$$\Rightarrow (t + 4)(t - 3) = 0$$

$$\Rightarrow t = 3 \quad \text{or} \quad -4$$

NOTES

Now $t = 3 \Rightarrow 3x^2 - 4x - 6 = 9 \Rightarrow 3x^2 - 4x - 15 = 0$

$\Rightarrow 3x^2 - 9x + 5x - 15 = 0$

$\Rightarrow (x - 3)(3x + 5) = 0$

$\Rightarrow x = 3 \text{ or } -\frac{5}{3}$

Also, $t = -4 \Rightarrow 3x^2 - 4x - 6 = 16 \Rightarrow 3x^2 - 4x - 22 = 0$

$$\Rightarrow \quad x = \frac{4 \pm \sqrt{16 - 4.3(-22)}}{6}$$

$$= \frac{4 \pm \sqrt{16 + 264}}{6} = \frac{4 \pm \sqrt{280}}{6}$$

$$\Rightarrow \quad x = \frac{2 \pm \sqrt{70}}{3}$$

Hence, $x = 3, -\frac{5}{3}, \frac{2 \pm \sqrt{70}}{3}$.

Example 4.20: Solve $\dfrac{\sqrt{1+x^2} + \sqrt{1-x^2}}{\sqrt{1+x^2} - \sqrt{1-x^2}} = 3$.

Solution: Simplifying given equation, we get

$$\sqrt{1+x^2} + \sqrt{1-x^2} = 3\sqrt{1+x^2} - 3\sqrt{1-x^2} \Rightarrow 2\sqrt{1+x^2} = 4\sqrt{1-x^2}$$

$$\Rightarrow \sqrt{1+x^2} = 2\sqrt{1-x^2}$$

$$\Rightarrow 1 + x^2 = 4(1 - x^2)$$

$$\Rightarrow 5x^2 = 3$$

$$\Rightarrow x^2 = \frac{3}{5}$$

$$\Rightarrow x = \pm\sqrt{\frac{3}{5}}.$$

CHECK YOUR PROGRESS

Solve the following equations:

1. $x^2 - 8x - 48 = 0$
2. $3x^2 + 10x + 3 = 0$
3. $\dfrac{77(x-2)}{x+3} - 21x = 7 + \dfrac{7x - 49}{x+3}$
4. $\dfrac{15x^2 - 16}{4} - 7x - 3$

NOTES

4.5 RELATION OF THE ROOTS

4.5.1 Symmetric Expression of Roots of a Quadratic Equation

We first prove that a quadratic equation cannot have more than two roots.

Suppose α, β, γ are three roots of $ax + bx + c = 0$.

Since α is a root of $ax^2 + bx + c = 0$, $x - \alpha$ is a factor of $ax^2 + bx + c$ (by Remainder Theorem). Similarly, $x - \beta$ and $x - \gamma$ are factors of $ax^2 + bx + c$.

So, $ax^2 + bx + c = k(x - \alpha)(x - \beta)(x - \gamma)$, where k is a non-zero constant. Now left hand side is a polynomial of degree 2, whereas right hand side is a polynomial of degree 3 but two polynomials can be equal only when their degrees are equal and coefficients of equal powers of x are equal. Thus, we have a contradiction. Hence, a quadratic equation cannot have more than two roots.

Definition: An equation $f(x) = 0$ is called an identity if it is satisfied by all values of x.

For example, $(x - 2)^2 - x^2 + 4x - 4 = 0$ is an identity while $x^2 - 6x + 5 = 0$ is not an identity, as $x = 2$ does not satisfy $x^2 - 6x + 5 = 0$ $(2^2 - 6.2 + 5 = 9 - 12 = -3 \neq 0)$.

Notation: An identity is denoted by $\equiv$.

$$\text{Thus, } (x - 2)^2 - x^2 + 4x - 4 \equiv 0.$$

Theorem. $ax^2 + bx + c = 0$ is an identity if and only if $a = b = c = 0$.

Proof: In case $a = b = c = 0$, given equation reduces to

$$0.x^2 + 0.x + 0 = 0$$

Which is clearly satisfied by all values of x.

Conversely, let $ax^2 + bx + c = 0$ be satisfied by all values of x. Take three distinct numbers α, β, γ. In particular, the given equation must be satisfied by $x = \alpha$, β and γ.

i.e.,

$$a\alpha^2 + b\alpha + c = 0 \quad \text{...(4.1)}$$

$$a\beta^2 + b\beta + c = 0 \quad \text{...(4.2)}$$

$$a\gamma^2 + b\gamma + c = 0 \quad \text{...(4.3)}$$

Equations (4.1) and (4.2) give, on subtraction

$$a(\alpha^2 - \beta^2) + b(\alpha - \beta) = 0$$

$$\Rightarrow \quad a(\alpha + \beta) + b = 0 \text{ as } \alpha \neq \beta. \quad \text{...(4.4)}$$

Further, equations (4.2) and (4.3) yield, on subtraction

$$a(\beta^2 - \gamma^2) + b(\beta - \gamma) = 0$$

$$\Rightarrow \quad a(\beta + \gamma) + b = 0 \text{ as } \beta \neq \gamma. \quad \text{...(4.5)}$$

Subtract equation (4.5) from equation (4.4) to get $a(\alpha - \gamma) = 0$

$$\Rightarrow a = 0 \quad \text{as } \alpha \neq \gamma$$

Then equation (4.4) $\Rightarrow b = 0$ and (1) $\Rightarrow c = 0$.

Hence, equation the theorem follows.

NOTES

Coming back to quadratic equation $ax^2 + bx + c = 0$ (i.e., $\alpha \neq 0$), we see that if α, β are its roots, then

$$ax^2 + bx + c \equiv a(x - \alpha)(x - \beta)$$

$$(a - a)x^2 + [b + a(\alpha + \beta)]x + c - a\alpha\beta \equiv 0$$

By previous theorem this means

$$b = -a(\alpha + \beta) \quad \text{or} \quad \alpha + \beta = -\frac{b}{a}$$

And $$c = a\alpha\beta \quad \text{or} \quad \alpha\beta = \frac{c}{a}$$

Thus, we get the following relation between roots and coefficients

$$\alpha + \beta = -\frac{b}{a}$$

$$\alpha\beta = \frac{c}{a}$$

***Aliter*:** The above relations can also be found as under:

Take, $$\alpha = \frac{-b + \sqrt{b^2 - 4ac}}{2a} \quad \text{and} \quad \beta = \frac{-b - \sqrt{b^2 - 4ac}}{2a}$$

Then, $$\alpha + \beta = \frac{-2b}{2a} = \frac{-b}{a}$$

And $$\alpha\beta = \frac{b^2 - (b^2 - 4ac)}{4a^2} = \frac{4ac}{4a^2} = \frac{c}{a}$$

The above relations imply that

$$ax^2 + bx + c = 0$$

is same as $$ax - a(\alpha + \beta)x + a\alpha\beta = 0 \text{ or}$$

$$x^2 - (\alpha + \beta)x + \alpha\beta = 0 \quad \text{as} \quad a \neq 0$$

This gives us the method of construction of a quadratic equation whose roots are given.

The required quadratic will be

x^2 – (sum of two roots)x + (product of two roots) = 0

Definition: Any expression involving α and β is called a symmetric function of α and β, if it remains unchanged when α and β are interchanged.

For instance, $\alpha^2 + \beta^2$, $\alpha\beta$, $\dfrac{\alpha^3 + \beta^3}{\alpha\beta}$, $\alpha^2\beta + \alpha\beta^2$

are all symmetric functions of α and β, where $\alpha^3 - \beta$ is not a symmetric function since in general $\alpha^3 - \beta$ need not be equal to $\beta^3 - \alpha$.

For example, if $\alpha = 1$, $\beta = 2$; $\alpha^3 - \beta = 1 - 2 = -1$ while $\beta^3 - \alpha = 8 - 1 = 7$.

With the help of relation $\alpha + \beta = -\frac{b}{a}$ and $\alpha\beta = \frac{c}{a}$ we can evaluate symmetirc function of α and β. The method is best illustrated with the help of examples.

Example 4.21:

(i) If α and β are the roots *of* $x^2 - px + q = 0$, form an equation whose roots are $\alpha\beta + \alpha + \beta$ *and* $\alpha\beta - \alpha - \beta$.

(ii) If α and β are the roots of $2x^2 - 4x + 1 = 0$, form the equation whose roots are $\alpha^2 + \beta$ and $\beta^2 + \alpha$.

NOTES

Solution:

(i) Let $\gamma = \alpha\beta + \alpha + \beta$, and $\delta = \alpha\beta - \alpha - \beta$

The equation whose roots are γ and δ is

$$x^2 - (\gamma + \delta)x + \gamma\delta = 0$$

Now α, β are roots of $x^2 - px + q = 0$

Implies that, $\alpha + \beta = -\frac{(-p)}{1} = p$

And $\alpha\beta = \frac{q}{1} = q$

This further yields that

$$\gamma = \alpha\beta + \alpha + \beta = q + p, \text{ and } \delta = \alpha\beta - \alpha - \beta = q - p.$$

Hence, the required equation is

$$x^2 - (q + p + q - p)x + (q + p)(q - p) = 0$$

i.e., $x^2 - 2qx + q^2 - p^2 = 0$

(ii) $\alpha + \beta = \frac{4}{2} = 2$

$$\alpha\beta = \frac{1}{2}$$

Now,
$$\begin{aligned} s_1 &= \alpha^2 + \beta + \beta^2 + \alpha \\ &= \alpha^2 + \beta^2 + \alpha + \beta \\ &= (\alpha + \beta)^2 - 2\alpha\beta + \alpha + \beta \\ &= 4 - 1 + 2 = 5 \end{aligned}$$

And
$$\begin{aligned} s_2 &= (\alpha^2 + \beta)(\beta^2 + \alpha) \\ &= \alpha^2\beta^2 + \alpha^3 + \beta^3 + \alpha\beta \\ &= \frac{1}{4} + \frac{1}{2} + (\alpha + \beta)^3 - 3\alpha\beta(\alpha + \beta) \\ &= \frac{3}{4} + 8 - \frac{3}{2}(2) \\ &= 5 + \frac{3}{4} = \frac{23}{4}. \end{aligned}$$

Hence, the required equation is

$$x^2 - 5x + \frac{23}{4} = 0$$

Or $4x^2 - 20x + 23 = 0.$

NOTES

Example 4.22: α, β are roots of $2x^2 + 3x + 7 = 0$. Find the values of $\frac{\alpha}{\beta} + \frac{\beta}{\alpha}$ and $\alpha^3 + \beta^3$.

Solution: $\alpha + \beta = -\frac{3}{2}$ and $\alpha\beta = \frac{7}{2}$

Now, $$\frac{\alpha}{\beta} + \frac{\beta}{\alpha} = \frac{\alpha^2 + \beta^2}{\alpha\beta} = \frac{(\alpha+\beta)^2 - 2\alpha\beta}{\alpha\beta}$$

$$= \frac{\frac{9}{4} - 2\cdot\frac{7}{2}}{\frac{7}{2}}$$

$$= \frac{\frac{9}{4} - 7}{\frac{7}{2}} = \frac{2}{7}\left(\frac{-19}{4}\right) = \frac{-19}{14}$$

Again, $$\alpha^3 + \beta^3 = (\alpha + \beta)(\alpha^2 - \alpha\beta + \beta^2)$$

$$= (\alpha + \beta)[(\alpha + \beta)^2 - 3\alpha\beta]$$

$$= \left(-\frac{3}{2}\right)\left(\frac{9}{4} - 3\cdot\frac{7}{2}\right)$$

$$= \left(-\frac{3}{2}\right)\left(\frac{9}{4} - \frac{21}{2}\right)$$

$$= \left(-\frac{3}{2}\right)\left(\frac{-33}{4}\right) = \frac{99}{8}.$$

Example 4.23: If α, β are the roots of $ax^2 + bx + c = 0$, find the equation with roots $a\alpha + b\beta$ and $b\alpha + a\beta$.

Solution: We have

$$(a\alpha + b\beta) + (b\alpha + a\beta) = (a + b)\alpha + (a + b)\beta$$

$$= (a + b)\alpha + (a + b)\beta$$

$$= (a + b)(\alpha + \beta)$$

$$= (a+b)\left(-\frac{b}{a}\right) = -\frac{b}{a}(a+b)$$

Also, $$(a\alpha + b\beta)\,(b\alpha + a\beta) = ab\alpha^2 + ab\beta^2 + (a^2 + b^2)\alpha\beta$$

$$= ab(\alpha^2 + \beta^2) + (a^2 + b^2)\alpha\beta$$

$$= ab\,[(\alpha + \beta)^2 - 2\alpha\beta] + (a^2 + b^2)\alpha\beta$$

$$= ab\left(\frac{b^2}{a^2} - \frac{2c}{a}\right) + (a^2 + b^2)\frac{c}{a}$$

$$= \frac{b(b^2 - 2ac)}{a} + \frac{(a^2 + b^2)c}{a}$$

$$= \frac{b(b^2 - 2ac) + (a^2 + b^2)c}{a}$$

Hence, the required equation is

NOTES

$$x^2 + \frac{b}{a}(a+b)x + \frac{b(b^2-2ac)+(a^2+b^2)c}{a} = 0$$

Or, $ax^2 + b(a + b)x + b(b^2 - 2ac) + (a^2 + b^2)c = 0$

Or, $ax^2 + b^2x + abx + b^3 + c(a^2 + b^2 - 2ab) = 0$

Or, $(ax + b^2)x + b(ax + b^2) + c(a - b)^2 = 0$

$\Rightarrow$ $(ax + b^2)(x + b) + c(a - b)^2 = 0$

Example 4.24: If the roots of $ax^2 + bx + c = 0$ are in the ratio $p : q$, prove that $ac(p + q)^2 = b^2pq$.

Solution: We know that $\alpha + \beta = -\frac{b}{a}$ and $\alpha\beta = \frac{c}{a}$

By hypothesis, $\frac{\alpha}{\beta} = \frac{p}{q}$

We are to eliminate α, β from these three relations.

$$\alpha\beta = \frac{c}{a} \text{ and } \frac{\alpha}{\beta} = \frac{p}{q} \Rightarrow \alpha^2 = \frac{pc}{aq}$$

And $$\frac{\alpha\beta}{\alpha/\beta} = \left(\frac{c}{a}\right)\Big/\frac{p}{q} \Rightarrow \beta^2 = \frac{cq}{ap}$$

Again, $$\alpha + \beta = -\frac{b}{a} \Rightarrow \alpha^2 + \beta^2 + 2\alpha\beta = \frac{b^2}{a^2}$$

So, we get

$$\frac{pc}{aq} + \frac{cq}{ap} + \frac{2c}{a} = \frac{b^2}{a^2}$$

Or, $$\frac{(p^2+q^2)c + 2cpq}{apq} = \frac{b^2}{a^2} \Rightarrow a(p^2 + q^2)c + 2capq = b^2pq$$

$$\Rightarrow ac(p^2 + q^2 + 2pq) = b^2pq$$

$$\Rightarrow ac(p + q)^2 = b^2pq.$$

Example 4.25: If α and β are the roots of $ax^2 + bx + c = 0$, form the equation whose roots are $\alpha^2 + \beta^2$ *and* $\alpha^{-2} + \beta^{-2}$.

Solution: Here, $\alpha + \beta = -\frac{b}{a}$ and $\alpha\beta = \frac{c}{a}$

Now $$\alpha^2 + \beta^2 = (\alpha + \beta)^2 - 2\alpha\beta = \frac{b^2}{a^2} - \frac{2c}{a} = \frac{b^2 - 2ac}{a^2}$$

And $$\alpha^{-2} + \beta^{-2} = \frac{1}{\alpha^2} + \frac{1}{\beta^2} = \frac{\alpha^2 + \beta^2}{\alpha^2\beta^2} = \frac{\frac{b^2-2ac}{a^2}}{\frac{c^2}{a^2}} = \frac{b^2 - 2ac}{c^2}$$

So, the sum of new roots $= \frac{b^2 - 2ac}{a^2} + \frac{b^2 - 2ac}{c^2} = \frac{(b^2 - 2ac)(c^2 + a^2)}{c^2a^2}$

And product of new roots $= \frac{(b^2 - 2ac)^2}{a^2c^2}$

NOTES

Hence, the required equation is

$$x^2 - \frac{(b^2 - 2ac)(c^2 + a^2)}{c^2a^2}x + \frac{(b^2 - 2ac)^2}{a^2c^2} = 0$$

Or, $$a^2c^2x^2 - (b^2 - 2ac)(c^2 + a^2)x + (b^2 - 2ac)^2 = 0$$

Example 4.26: For what values of m will the equation

$(m + 1)x^2 + 2(m + 3)x + (2m + 3) = 0$ have equal roots?

Solution: As shown in section 5.5, the equation will have equal roots, if and only if

$[2(m + 3)]^2 = 4(m + 1)(2m + 3)$

Or, $m^2 + 6m + 9 = 2m^2 + 5m + 3$

Or, $m^2 - m - 6 = 0$

Or, $(m - 3)(m + 2) = 0$ or $m = 3$ or -2.

Example 4.27: Show that $(x - a)(x - b) = h^2$ have real roots.

Solution: Given equation can be simplified to

$$x^2 - (a + b)x + ab - h^2 = 0$$

Discriminant, $= (a + b)^2 - 4(ab - h^2)$

$= (a + b)^2 - 4ab + 4h^2$

$= (a - b)^2 + 4h^2$

Which is always a positive quantity.

Hence, the roots of given equation are always real.

CHECK YOUR PROGRESS

5. If p, q are the roots of $3x^2 + 6x + 2 = 0$, form an equation whose roots are
$$\frac{-p^2}{q} \text{ and } \frac{-q^2}{p}$$
6. Find k if the roots of $2x^2 + 3x + k = 0$ are equal.
7. If α and β are the roots of $ax^2 + bx - c = 0$, form an equation whose roots are
$$\frac{1}{a\alpha + b}, \frac{1}{a\beta + b}$$
8. Form an equation whose roots are squares of the roots of $ax^2 + bx + c = 0$.

4.5.2 Simultaneous Equations in Two Unknowns

In this section we will consider solutions of linear and non-linear simultaneous equations in two unknowns. There are several techniques for solving such problems and we shall illustrate some of them with the help of examples.

NOTES

Example 4.28: Solve $4x - 3y = 1$, $12xy + 13x^2 = 25$

Solution: From $4x - 3y = 1$, $y = \frac{4x-1}{3}$

Substituting the value of y in $12xy + 13x^2 = 25$, we get

$$12x\left(\frac{4x-1}{3}\right) + 13x^2 = 25 \Rightarrow 16x^2 - 4x + 13x^2 = 25$$
$$\Rightarrow 29x^2 - 4x - 25 = 0$$
$$\Rightarrow (29x + 25)(x - 1) = 0$$
$$\Rightarrow x = 1 \quad \text{or} \quad -\frac{25}{29}$$

Then, $y = \frac{4x-1}{3}$ gives that either $y = 1$ or $-\frac{43}{29}$. Hence, the required solution is $x = 1$, $y = 1$

Or, $x = -\frac{25}{29}$, $y = -1\frac{14}{29}$.

Example 4.29: Solve $x^2 + y^2 = 185$, $x - y = 3$.

Solution: Now, $(x - y)^2 = x^2 + y^2 - 2xy \Rightarrow 9 = 185 - 2xy$

$$\Rightarrow 2xy = 176 \Rightarrow xy = 88$$

Again, $(x + y)^2 = (x - y)^2 + 4xy \Rightarrow (x + y)^2 = 9 + 352 = 361$

$$x + y = \pm 19$$

Taking +ve sign, we get

$$x - y = 3, x + y = 19 \text{ and so } x = 11, y = 8$$

Taking –ve sign, we get

$$x - y = 3, x + y = -19 \text{ and so } x = -8, y = -11$$

Hence, the required solution is $x = 11$, $y = 8$

Or, $x = -8$, $y = -11$

Example 4.30: Solve $2x + 3y = 5$, $xy = 1$.

Solution: Since $xy = 1$, we get $6xy = 6$, *i.e.*, $2x \,.\, 3y = 6$.

Now, we have $2x + 3y = 5$ and $2x \,.\, 3y = 6$

As $(2x - 3y)^2 = (2x + 3y)^2 - 4.2x.3y$, we get

$$(2x - 3y)^2 = 25 - 24 = 1 \quad \Rightarrow \quad 2x - 3y = \pm 1$$

Taking +ve sign, we obtain $2x + 3y = 5$, $2x - 3y = 1$

$\Rightarrow$ $x = \frac{3}{2}$, $y = \frac{2}{3}$.

Taking –ve sign, we obtain $2x + 3y = 5$, $2x - 3y = -1$

$\Rightarrow$ $x = 1$, $y = 1$.

Hence, required solution is $x = 1$, $y = 1$

$$x = 1\frac{1}{2}, \quad y = \frac{2}{3}.$$

NOTES

Example 4.31: Solve $x^3 + y^3 = 4914$, $x + y = 18$.

Solution. We know that $(x + y)^3 = x^3 + y^3 + 3xy(x + y)$

This gives that $(18)^3 = 4914 + 3xy \,.\, 18$

Or, $324 = 273 + 3xy \Rightarrow 108 = 91 + xy \Rightarrow xy = 17.$

Now, $x + y = 18, \quad xy = 17$

Solving by the method discussed in previous example, we get

$$x = 1, \quad y = 17 \quad \text{or} \quad x = 17, \quad y = 1$$

Example 4.32: Solve $x + y + \sqrt{xy} = 14$

$$x^2 + y^2 + xy = 84$$

Solution: $x^2 + y^2 + xy = (x + y)^2 - xy = (x + y + \sqrt{xy})(x + y - \sqrt{xy})$

$\Rightarrow \quad 84 = 14(x + y - \sqrt{xy}) \quad \Rightarrow \quad x + y - \sqrt{xy} = 6$

But, $x + y + \sqrt{xy} = 14$

Hence, $x + y = 10, \sqrt{xy} = 4 \quad \text{or} \quad xy = 16 \Rightarrow y = \dfrac{16}{x}$

Putting values of y in $x + y = 10$ we get, $x = 8, \; y = 2$

Or, $x = 2, \; y = 8.$

Example 4.33: (Homogenous Equations)

Solve $x^2 + xy + 4y^2 = 6$

$3x^2 + 8y^2 = 14$

(Note in such equations that sum of powers of x and y in each term is same.)

Solution: Both the equations can be re-written as

$$x^2\left(1 + \frac{y}{x} + \frac{4y^2}{x^2}\right) = 6 \text{ and } x^2\left(3 + \frac{8y^2}{x^2}\right) = 14.$$

Divide first equation by second equation and then put $\dfrac{y}{x} = m.$

Thus, we get $\dfrac{1 + m + 4m^2}{3 + 8m^2} = \dfrac{3}{7}$

$\Rightarrow \quad 28m^2 + 7m + 7 = 9 + 24m^2$

$\Rightarrow \quad 4m^2 + 7m - 2 = 0$

$\Rightarrow \quad (m + 2)(4m - 1) = 0$

$\Rightarrow \quad m = -2 \quad \text{or} \quad 1/4$

In first case, $y = -2x$, or substituting this value of y in first of the given equations, we get

$$x^2 - 2x^2 + 16x^2 = 6$$

Or, $15x^2 = 6 \quad \Rightarrow \quad x = \pm\sqrt{\dfrac{2}{5}}$

NOTES

Then, $\quad y = -2x \;\Rightarrow\; y = \mp 2\sqrt{\frac{2}{5}}$

In second case $4y = x$; on substituting this value of x in first of the given equations, we get $16y^2 + 4y^2 + 4y^2 = 6$.

Or, $\quad 24y^2 = 6 \;\Rightarrow\; y = \pm\frac{1}{2}$ Thus, $x = \pm 2$

Hence, the required solutions are

$$x = 2, \quad y = \frac{1}{2}; \quad x = -2, \quad y = -\frac{1}{2}$$

$$x = \sqrt{\frac{2}{5}}, \quad y = -2\sqrt{\frac{2}{5}}; \quad x = -\sqrt{\frac{2}{5}}, \quad y = 2\sqrt{\frac{2}{5}}.$$

Example 4.34: (Symmetrical Equations)

Solve
$$x^4 + y^4 = 257$$
$$x + y = 5$$

(In such equations, if we replace x with y and y with x, the equations are unchanged.)

Solution: Put $x = u + v$ and $y = u - v$.

By second given equation, we get $2u = 5$ or $u = 5/2$.

Hence, $\quad x = \frac{5}{2} + v, \qquad y = \frac{5}{2} - v$

Substituing these values in first of the given equation and recalling that

$(a + b)^4 = a^4 + 4a^3b + 6a^2b^2 + 4ab^3 + b^4$, we get

$$\left(\frac{5}{2} + v\right)^4 + \left(\frac{5}{2} - v\right)^4 = 257$$

$$2\left[\left(\frac{5}{2}\right)^4 + 6\left(\frac{5}{2}\right)^2 v^2 + v^4\right] = 257$$

$$\Rightarrow \quad 2\left(\frac{625}{16} + \frac{75}{2}v^2 + v^4\right) = 257$$

$$\Rightarrow \quad 625 + 600v^2 + 16v^4 = 2056$$

$$\Rightarrow \quad 16v^4 + 600v^2 - 1431 = 0$$

$$\Rightarrow \quad 16v^4 + 636v^2 - 36v^2 - 1431 = 0$$

$$\Rightarrow 4v^2(4v^2 + 159) - 9(4v^2 + 159) = 0$$

$$\Rightarrow \quad (4v^2 - 9)(4v^2 + 159) = 0$$

$$\Rightarrow \quad v^2 = \frac{9}{4} \quad \text{or} \quad v^2 = -\frac{159}{4}$$

Thus, $v = \pm\frac{3}{2}$ or $v = \pm\frac{\sqrt{-159}}{2}$

NOTES

Hence, $x = 4, \quad y = 1; \quad x = 1, \quad y = 4$

$$x = \frac{5+\sqrt{-159}}{2}, \quad y = \frac{5-\sqrt{-159}}{2}$$

$$x = \frac{5-\sqrt{-159}}{2}, \quad y = \frac{5+\sqrt{-159}}{2}$$

Example 4.35: Solve $x^2 + y^2 = 29$, $x - y = 3$.

Solution: $(x-y)^2 = x^2 + y^2 - 2xy \Rightarrow 9 = 29 - 2xy$

$\Rightarrow 2xy = 20$

$\Rightarrow xy = 10$

Now, $(x+y)^2 = x^2 + y^2 + 2xy$

$= 29 + 20 = 49$

$\Rightarrow \quad x + y = \pm 7$

Taking +ve sign, we get, $2x = 10 \Rightarrow x = 5$

and $y = x - 3 = 2$

Taking –ve sign, we get $2x = -4 \Rightarrow x = -2$

And $y = x - 3 = -5$

Hence, $x = -2, \quad y = -5$

Or, $x = 5, \quad y = 2$

Example 4.36: Solve $2x^2 + 3xy = 26, \quad 3y^2 + 2xy = 39$.

Solution: $2x^2 + 3xy = 26 \quad \Rightarrow \quad 2 + \frac{3y}{x} = \frac{26}{x^2}$

And $3y^2 + 2xy = 39 \quad \Rightarrow \quad \frac{3y^2}{x^2} + \frac{2y}{x} = \frac{39}{x^2}$

These equations give on division,

$$\frac{2+3m}{3m^2+2m} = \frac{26}{39} = \frac{2}{3}, \text{ where } m = \frac{y}{x}$$

$\Rightarrow \quad 6m^2 + 4m = 6 + 9m$

$\Rightarrow \quad 6m^2 - 5m - 6 = 0$

$\Rightarrow \quad 6m^2 - 9m + 4m - 6 = 0$

$\Rightarrow \quad (2m-3)(3m+2) = 0$

$\Rightarrow \quad m = \frac{3}{2} \quad \text{or} \quad -\frac{2}{3}$

In case, $m = \frac{3}{2}, \quad y = mx = -\frac{3}{2}x,$

So, $2x^2 + 3xy = 26$

$\Rightarrow \quad 2x^2 + \frac{9x^2}{2} = 26$

$\Rightarrow \quad x^2 = 4 \quad \Rightarrow \quad x = \pm 2$

Then, $y = \pm 3$

In case $m = -\frac{2}{3}, \quad y = mx = -\frac{2}{3}x$

So, $2x^2 + 3xy = 26$

$\Rightarrow$ $2x^2 - 2x^2 = 26$

$\Rightarrow$ $0 = 26$, which is absurd.

Hence the only admissible value of m is $\frac{3}{2}$ and then the roots are

$$x = 2, \quad y = 3$$
$$x = -2, \quad y = -3.$$

Example 4.37: By selling a table for Rs 56, gain is as much per cent as its cost in rupees. What is the cost price?

Solution: Let the cost price be x and gain be y

Then, $x + y = 56$

$$\text{Percentage gain} = \frac{100y}{x}$$

This is equal to x

i.e., $100y = x^2$

So, we are to solve

$$x + y = 56$$
$$100y = x^2$$

First equation gives $y = 56 - x$

Second equation then reduces to

$$x^2 = 5600 - 100x$$

Or, $x^2 + 100x - 5600 = 0$

$(x + 140)(x - 40) = 0 \quad \Rightarrow \quad x = 40 \quad \text{or} \quad -140$

As x is cost, it must be a positive quantity.

Hence, $x = 40$

So, cost price of table is Rs 40.

Example 4.38: If the Demand and Supply Laws are respectively given by the equation

$$4q + 9p = 48 \quad and \quad p = \frac{q}{9} + 2$$

Find the equilibrium price and quantity.

Solution: In equilibrium, demand = supply, i.e., in the above two equations p, q stand for same quantities.

We are to solve

$$4q + 9p = 48$$
$$p = \frac{q}{9} + 2$$

NOTES

In first equation, substitution of

$$p = \frac{q}{9} + 2$$

Gives $\quad 4q + q + 18 = 48$

$\Rightarrow \quad 5q = 30 \quad \Rightarrow \quad q = 6$

Then, $\quad p = \frac{q}{9} + 2 = \frac{2}{3} + 2 = \frac{8}{3}$

Hence, price is $\frac{8}{3}$ and quantity is 6.

Example 4.39: Solve $\sqrt{\frac{x}{y}} + \sqrt{\frac{y}{x}} = \frac{5}{2}$

$$x + y = 10$$

Solution: $\quad \sqrt{\frac{x}{y}} + \sqrt{\frac{y}{x}} = \frac{5}{2}$

$\Rightarrow \quad x + y = \frac{5}{2}\sqrt{xy}$

$\Rightarrow \quad 10 = \frac{5}{2}\sqrt{xy} \quad \Rightarrow \quad xy = 16$

Then, $\quad (x - y)^2 = (x + y)^2 - 4xy$

$$= 100 - 64 = 36$$

$\Rightarrow \quad x - y = \pm 6$

Taking +ve sign and solving with $x + y = 10$, we get

$$x = 8 \text{ and } y = 2$$

Taking –ve sign and solving with $x + y = 10$, we obtain

$$x = 2 \quad \text{and} \quad y = 8.$$

Example 4.40: As the number of units manufactured increases from 4000 to 6000, the total cost of production increases from Rs 22,000 to Rs. 30,000. Find the relationship between the cost (y) and the number of units made (x), if the relationship is linear.

Solution: Let the relationship between x and y be given by

$$ax + by + c = 0$$

When $x = 4000$, $y = 20,000$

So, $\quad 4000a + 22,000\, b + c = 0 \quad$ ...(1)

Also when, $x = 6,000$, $y = 30,000$

So, $\quad 6000a + 30000b + c = 0 \quad$ ...(2)

Multiply equation (1) by 3 and equation (2) by 2 to obain

$$12000a + 66000b + 3c = 0$$

$$12000a + 60000b + 2c = 0$$

On subtraction, we get

$$6000\, b + c = 0$$

Or $\qquad b = -\frac{c}{6000}$

Then, equation (1) implies $4000a - \frac{11c}{3} + c = 0$

Or, $\qquad 4000\,a = \frac{8c}{3}$

$\Rightarrow \qquad a = \frac{c}{1500}$

Thus, the linear relation is

$$\frac{c}{1500}x - \frac{c}{6000}y + c = 0$$

Or $\qquad 4x - y + 6000 = 0$

Or $\qquad y = 4x + 6000.$

4.5.3 Simultaneous Equations in Three or More than Three Unknowns

As in the case of two unkowns, there is no fixed method to solve general non-linear simultaneous equations in three unknowns, however, there are methods for solving particulars types of such equations, we shall illustrate some of them by examples.

Note: Cross-multiplication method is applicable only when at least two of the given equations are of the type

$$a_1x + b_1y + c_1z = 0 \qquad ...(4.6)$$

And $\qquad a_2x + b_2y + c_2z = 0 \qquad ...(4.7)$

Multiplying first equation by b_2 and second by b_1 and subtracting the resulting second equation from resulting first equation,

we get $\qquad (a_1b_2 - a_2b_1)x = -(c_1b_2 - c_2b_1)z$

Or $\qquad \frac{x}{b_1c_2 - b_2c_1} = \frac{z}{a_1b_2 - a_2b_1}$

Similarly, eliminating x from equations (4.6) and (4.7), we get

$$\frac{y}{c_1a_2 - c_2a_1} = \frac{z}{a_1b_2 - a_2b_1}$$

Thus, $\qquad \frac{x}{b_1c_2 - b_2c_1} = \frac{y}{c_1a_2 - c_2a_1} = \frac{z}{a_1b_2 - a_2b_1}.$

Example 4.41: Solve

$$5x - 4y + z = 0$$
$$2x + 5y - 4z = 0$$
$$x^2 - 2y^2 + z^2 = 0.$$

Solution: By cross-multiplication,

$$\frac{x}{16-5} = \frac{y}{2-(-20)} = \frac{z}{25-(-8)}$$

NOTES

$$\Rightarrow \qquad \frac{x}{11} = \frac{y}{22} = \frac{z}{33} \quad \Rightarrow \quad x = \frac{y}{2} = \frac{z}{3} = k \quad \text{(say)}$$

Thus, $x = k$, $y = 2k$, $z = 3k$.

Substituting these values in third equation, we get

$$k^2 - 8k^2 + 9k^2 = 0 \quad \text{or} \quad 2k^2 = 0 \quad \Rightarrow \quad k = 0$$

Hence, $x = 0$, $y = 0$, $z = 0$.

Example 4.42: Solve $3x + y - 2z = 0$, $4x - y - 3z = 0$

and $x^3 + y^3 + z^3 = 467$.

Solution: By cross-multiplication

$$\frac{x}{-3-2} = \frac{y}{-8-(-9)} = \frac{z}{-3-4}$$

Or, $$\frac{x}{-5} = \frac{y}{1} = \frac{z}{-7}$$

Or, $$\frac{x}{5} = \frac{y}{-1} = \frac{z}{7} = k \quad \text{(say)}$$

Thus, $x = 5k$, $y = -k$, $z = 7k$.

On substitution of these values in third equation, we obtain

$$125k^3 - k^3 + 343k^3 = 467$$

$$467k^3 = 467 \quad \text{or} \quad k^3 = 1 \quad \Rightarrow k = 1$$

Hence, $x = 5$, $y = -1$, $z = 7$.

Example 4.43: Solve
$$x^2 + xy + y^2 = 13$$
$$y^2 + yz + z^2 = 49$$
$$z^2 + zx + x^2 = 31.$$

Solution: Subtract second equation from first to obtain

$$(x^2 - z^2) + y(x - z) = -36$$

Or, $$(x - z)(x + y + z) = -36 \qquad \text{...(1)}$$

Similarly, subtracting third equation from second equation, we obtain

$$(y - x)(x + y + z) = 18 \qquad \text{...(2)}$$

Divide equation (1) by equation (2) to get $\frac{x-z}{y-x} = -2$

Or, $$x - z = -2y + 2x$$

Or, $$2y = x + z \quad \Rightarrow \quad y = \frac{x+z}{2}$$

Substitute this value in the given second equation, we get

$$\frac{(x+z)^2}{4} + \frac{z(x+z)}{2} + z^2 = 49$$

Or, $$x^2 + z^2 + 2xz + 2zx + 2z^2 + 4z^2 = 196$$

In other words, $$x^2 + 4xz + 7z^2 = 196$$

NOTES

Also, we are given that $x^2 + xz + z^2 = 31$

Thus, we obtain $$\frac{x^2 + 4xz + 7z^2}{x^2 + xz + z^2} = \frac{196}{31}$$

Or, $$\frac{1 + 4z/x + 7z^2/x^2}{1 + z/x + z^2/x^2} = \frac{196}{31}$$

Put $z/x = m$

This gives that $31(7m^2 + 4m + 1) = 196(1 + m + m^2)$

$$21m^2 - 72m - 165 = 0$$

$\Rightarrow$ $7m^2 - 24m - 55 = 0$

$\Rightarrow$ $(m - 5)(7m + 11) = 0$

$\Rightarrow$ $m = 5$ or $m = -11/7$

In case $m = 5$, $z = 5x$, so $y = \dfrac{z + x}{2} = 3x$

From first of the given equations, we get

$$x^2 + 3x^2 + 9x^2 = 13 \Rightarrow 13x^2 = 13 \Rightarrow x = \pm 1$$

So, $y = \pm 3$ and $z = \pm 5$.

In case $m = -\dfrac{11}{7}$, $z = -\dfrac{11}{7}x$, and so $y = \dfrac{z + x}{2} = -\dfrac{2}{7}x$.

Again, with the help of first of the given equations, we get

$$x^2 - \frac{2x^2}{7} + \frac{4}{49}x^2 = 13$$

Or $39x^2 = 637 \Rightarrow x = \pm\dfrac{7}{\sqrt{3}}$

Then, $y = \mp\dfrac{2}{\sqrt{3}}$ and $z = \mp\dfrac{11}{\sqrt{3}}$

Hence, the complete solution is $x = 1, y = 3, z = 5$.

Or, $x = -1$, $y = -3$, $z = -5$

Or, $x = \dfrac{7}{\sqrt{3}}$, $y = \dfrac{-2}{\sqrt{3}}$, $z = \dfrac{-11}{\sqrt{3}}$

Or, $x = -\dfrac{7}{\sqrt{3}}$, $y = \dfrac{2}{\sqrt{3}}$, $z = \dfrac{11}{\sqrt{3}}$.

Example 4.44: Solve
$$x^2 + xy + xz = 45$$
$$y^2 + yz + yx = 75$$
$$z^2 + zx + zy = 105.$$

Solution: The given equations are equivalent to

$$x(x + y + z) = 45$$
$$y(x + y + z) = 75$$
$$z(x + y + z) = 105$$

Adding, we get $(x + y + z)^2 = 225$

i.e., $x + y + z = \pm 15$

Hence, $x = \pm 3$, $y = \pm 5$, $z = \pm 7$.

So, the required solutions are $x = 3$, $y = 5$, $z = 7$

Or, $x = -3$, $y = -5$, $z = -7$.

NOTES

Example 4.45: Solve
$$xyz = 231$$
$$xyu = 420$$
$$xzu = 660$$
$$yzu = 1540.$$

Solution: Multiplying all the given equations, we get

$$x^3y^3z^3u^3 = 231 \times 420 \times 660 \times 1540$$
$$= 3 \times 7 \times 11 \times 2^2 \times 3 \times 5 \times 7 \times 2^2 \times 3 \times 5 \times 11 \times 2^2 \times 5 \times 7 \times 11$$
$$= 2^6 \times 3^3 \times 5^3 \times 7^3 \times 11^3$$

Thus, $xyzu = 2^2 \times 3 \times 5 \times 7 \times 11$
$$= 4620$$

Dividing in turn by 1st, 2nd, 3rd and 4th equations, we get
$$u = 20, \quad z = 11, \quad y = 7, \quad x = 3.$$

Hence, $x = 3$, $y = 7$, $z = 11$ and $u = 20$.

Example 4.46: Solve $x + y + z = 12$
$$x^2 + y^2 + z^2 = 50$$
And $x^3 + y^3 + z^3 = 216.$

Solution: Now, $(x + y + z)^2 = x^2 + y^2 + z^2 + 2(xy + yz + zx)$
$$\Rightarrow 144 = 50 + 2(xy + yz + zx)$$
$$\Rightarrow xy + yz + zx = 47$$

Further, $x^3 + y^3 + z^3 - 3xyz = (x + y + z)(x^2 + y^2 + z^2 - xy - yz - zx)$

gives that $216 - 3xyz = 12(50 - 47) = 36 \Rightarrow 3xyz = 180$ or $xyz = 60$

Thus, we have
$$x + y + z = 12 \qquad ...(1)$$
$$xy + yz + zx = 47 \qquad ...(2)$$
And
$$xyz = 60 \qquad ...(3)$$

From equation (3), $yz = \frac{60}{x}$ and from equation (1), $y + z = 12 - x$

Substituting these values in equation (2), we get

$$\frac{60}{x} + x(12 - x) = 47$$
$$60 + 12x^2 - x^3 = 47x$$
$$x^3 - 12x^2 + 47x - 60 = 0$$

i.e., $(x - 3)(x^2 - 9x + 20) = 0$
$$(x - 3)(x - 4)(x - 5) = 0$$

$\therefore$ $x = 3$ or 4 or 5

For $x = 3$, $yz = 20$, $y + z = 9$ $\Rightarrow$ $y = 5$, $z = 4$; $y = 4$, $z = 5$

For $x = 4$, $yz = 15$, $y + z = 8$ $\Rightarrow$ $y = 3$, $z = 5$; $y = 5$, $z = 3$

For $x = 5$, $yz = 12$, $y + z = 7$ $\Rightarrow$ $y = 3$, $z = 4$; $y = 4$, $z = 3$

Hence, the complete solution is

$$x = 3, y = 4, z = 5$$
$$x = 3, y = 5, z = 4$$
$$x = 4, y = 3, z = 5$$
$$x = 4, y = 5, z = 3$$
$$x = 5, y = 3, z = 4$$

And $\quad x = 5, y = 4, z = 3$

NOTES

Example 4.47: Solve $xy + x + y = 23$

$$xz + x + z = 41$$
$$yz + y + z = 27.$$

Solution: The given equations are equivalent to

$$xy + x + y + 1 = 24 \quad \text{i.e.,} \quad (x + 1)(y + 1) = 24 \qquad ..(1)$$
$$xz + x + z + 1 = 42 \quad \text{i.e.,} \quad (x + 1)(z + 1) = 42 \qquad ...(2)$$

And $\quad yz + y + z + 1 = 28 \quad \text{i.e.,} \quad (y + 1)(z + 1) = 28 \qquad ...(3)$

Multiplying equations (1), (2) and (3), we obtain

$$(x + 1)^2(y + 1)^2(z + 1)^2 = 24\times42\times28$$
$$= 6 \times 4 \times 6 \times 7 \times 4 \times 7$$

$\Rightarrow \quad (x + 1)(y + 1)(z + 1) = \pm 6 \times 4 \times 7 = \pm 168$

Dividing successively by equations (1), (2) and (3), we get

$$z + 1 = \pm 7, y + 1 = \pm 4 \text{ and } x + 1 = \pm 6$$

In other words $x = 5$ or -7, $y = 3$ or -5 and $z = 6$ or -8

Thus, $\quad x = 5, \quad y = 3, \quad z = 6$

Or, $\quad x = -7, \quad y = -5, \quad x = -8.$

CHECK YOUR PROGRESS

Solve the following equations:

9. $5x - y = 3$, $\quad y^2 - 6x^2 = 25$
10. $3x + 4y = 18$, $\dfrac{1}{x} + \dfrac{1}{y} = \dfrac{5}{6}$
11. $3x - 5y = 2$, $\quad xy = 8$
12. $x + y = 30$, $xy = 216$
13. $x - y = -18$, $xy = 1363$
14. $5x + 2y = 8$, $\quad 9x - 5y = 23$

4.6 SUMMARY

NOTES

In this unit, you have learned that:

- An equation of degree 2 is called a quadratic equation.
- A quadratic equation is called pure if it does not contain single power of x. In other words, in a pure quadratic equation, coefficient of x must be zero. Thus, a pure quadratic equation is of the type $ax^2 + b = 0$ with $a \neq 0$.
- If the expression $ax^2 + bx + c$ can be factored into linear factors, then each of the factors, put to zero, provides us with a root of the given quadratic equation. Thus, if $ax^2 + bx + c = a(x - a)(x - b)$, then the roots of $ax^2 + bx + c = 0$ are a and b.
- An equation $f(x) = 0$ is called an identity if it is satisfied by all values of x.

4.7 KEY TERMS

- **Quadratic equation:** It is an equation of degree 2.
- **Reciprocal equation:** It is an equation like $x^4 + x^3 - 4x^2 + x + 1 = 0$, where the terms are arranged according to the descending powers of x, the coefficients of terms equidistant from first and last term is equal or differ in sign.

4.8 ANSWERS TO 'CHECK YOUR PROGRESS'

1. $x^2 - 8x - 48=0$

$\Rightarrow$ $x^2 - 12x + 4x - 48 = 0$

$x(x - 12) + 4(x - 12) = 0$

$(x - 12)\ (x + 4) = 0$

$x = 12, -4$

2. $3x^2 + 10x + 3 = 0$

$\Rightarrow$ $3x^2 + 9x + x + 3 = 0$

$\Rightarrow$ $3x(x + 3) + 1(x + 3) = 0$

$\Rightarrow$ $(x + 3)(3x + 1) = 0$

$x = -3 \text{ or } -\dfrac{1}{3}$

3. The given expression

$$\frac{77(x-2)}{x+3} - 21x = 7 + \frac{7x-49}{x+3}$$

$$77(x - 2) - 21x(x + 3) = 7(x + 3) + 7x - 49$$

$$77x - 154 - 21x^2 - 63x = 7x + 21 + 7x - 49$$

$$21x^2 + 14x + 63x - 77x + 21 - 49 + 154 = 0$$

$$21x^2 + 126 = 0 \Rightarrow x^2 + 6 = 0$$

$x = \pm\sqrt{-6}$ roots are imaginary

4. $$\frac{15x^2-16}{4} = 7x - 3$$

$$\Rightarrow \quad 15x^2 - 28x - 12$$

$$\Rightarrow \quad 15x^2 - 16 - 28x - 4 = 0$$

$$\Rightarrow \quad 15x^2 - 30x + 2x - 4 = 0$$

$$15x(x -2) + 2(x - 2) = 0$$

$$(x - 2)\ (15x + 2) = 0$$

$$x = 2 \text{ or } - \frac{2}{15}$$

5. p an q are roots of equation $3x^2 + 6x + 2 = 0$

$$\Rightarrow \quad p + q = -\frac{6}{3} = -2 \text{ and } pq = \frac{2}{3}$$

Roots of new equation are:

$$-\frac{p^2}{q} \text{ and } \frac{-q^2}{p}$$

$$\text{sum of roots} = \frac{-p^2}{q} - \frac{q^2}{p} = \frac{-p^3-q^3}{pq} = -\frac{p^3+q^3}{pq}$$

$$= -\left\{\frac{(p+q)^3-3pq(p+q)}{pq}\right\}$$

$$= -\left\{\frac{(-2)^3-3\times\frac{2}{3}(-2)}{2/3}\right\}$$

$$= -\left\{\frac{-8+4}{2/3}\right\} = 6$$

NOTES

Product of roots = $-\dfrac{p^2}{q}\times\dfrac{-q^2}{p}$ $-pq=$ $\dfrac{2}{3}$

so, equation is

x^2 – (sum of roots) x + product of roots = 0

$x^2 - 6x + \dfrac{2}{3} = 0$

$3x^2 - 18x + 2 = 0$

6. If roots of equation $2x^2 + 3x + k = 0$ are equal, then its discriminant should be zero, i.e., $(3)^2 - 4 \times 2 \times k = 0$

$\Rightarrow \quad 9 - 8k = 0$

$k = \dfrac{9}{8}$

7. α and β are roots of equation $ax^2 + bx - c = 0$

So, $\alpha + \beta = -\dfrac{b}{a}$ and $\alpha\beta = -\dfrac{c}{a}$

If roots of equation are $\dfrac{1}{a\alpha+b}$ and $\dfrac{1}{a\beta+b}$

$$= \frac{a\beta+b+a\alpha+b}{(a\alpha+b)(a\beta+b)} = \frac{a(\alpha+\beta)+2b}{a^2(\alpha\beta)+ab\alpha+ab\beta+b^2}$$

$$= \frac{a(-b/a)+2b}{a^2(-c/a)+ab(\alpha+\beta)+b^2} = \frac{b}{-ac+ab\left(\dfrac{-b}{a}\right)+b^2}$$

$$= \frac{b}{-ac-b^2+b^2} = -\frac{b}{ac}$$

Product of roots $= \dfrac{1}{(a\alpha+b)(a\beta+b)} = \dfrac{1}{a^2\alpha\beta+ab(\alpha+\beta)+b^2}$

$$= \frac{1}{a^2\times\left(\dfrac{-c}{a}\right)+ab\left(-\dfrac{b}{a}\right)+b^2} = \frac{1}{-ac} = -\frac{1}{ac}$$

New equation is : $x^2 - \left(-\dfrac{b}{ac}\right)x - \dfrac{1}{ac} = 0$

$\Rightarrow \quad ac\,x^2 + bx - 1 = 0$

NOTES

8. Let roots of equation be α and β.

   Hence, $\alpha + \beta = -\frac{b}{a}$ and $\alpha\beta = \frac{c}{a}$

   If roots are α^2 and β^2

   Sum of roots $\alpha^2 + \beta^2 = (\alpha + \beta)^2 - 2\alpha\beta$

   $$= \frac{b^2}{a^2} - \frac{2c}{a} = \frac{b^2 - 2ac}{a^2}$$

   Product of roots $= \alpha^2\beta^2 = (\alpha\beta)2 = \frac{c^2}{a^2}$

   New equation is:

   $$x^2 - \left(\frac{b^2 - 2ac}{a^2}\right)x + \frac{c^2}{a^2} = 0$$

   $$a^2x^2 - (b^2 - 2ac)\,x + c^2 = 0$$

   $$\Rightarrow \quad a^2x^2 + (2ac - b^2)\,x + c^2 = 0$$

9. From given pair of equations:

   $5x - y = 3$ and $y^2 - 6x^2 = 25$

   $y = 5x - 3$

   Putting this value of y in second equation ,we get

   $(5x - 3)^2 - 6x^2 = 25$

   $25x^2 - 30x + 9 - 6x^2 - 25 = 0$

   $19x^2 - 30x - 16 = 0$

   $19x^2 - 38x + 8x - 16 = 0$

   $19x(x - 2) + 8(x - 2) = 0$

   $(x - 2)\ (19x + 8) = 0$

   Hence, $x = 2$ or $-\frac{8}{19}$

   and $y = 5 \times 2 - 3 = 7$ or $\frac{-5\times 8}{19} - 3 = \frac{-97}{19}$

   so, $x = 2,\ y = 7$ or $x = \frac{-8}{19},\ y = -\frac{97}{19}$

10. Given equations are:

$$3x + 4y = 18 \text{ and } \frac{1}{x}+\frac{1}{y}=\frac{5}{6}$$

NOTES

From second equation $\frac{1}{y} = \frac{5}{6}-\frac{1}{x} = \frac{5x-6}{6x}$

$\Rightarrow \quad y = \frac{6x}{5x-6}$

Putting this value of y in first equation, we get $3x+4\frac{6x}{5x-6}=18$

$\Rightarrow \quad 3x(5x-6) + 24x = 18(5x-6)$

$\Rightarrow \quad 15x^2 - 18x + 24x = 90x - 108$

$\Rightarrow \quad 15x^2 - 84x + 108 = 0$

$\Rightarrow \quad 5x^2 - 28x + 36 = 0$

$\Rightarrow \quad 5x^2 - 10x - 18x + 36 = 0$

$\Rightarrow \quad 5x(x-2) - 18(x-2) = 0$

$\Rightarrow \quad (x-2)(5x-18) = 0$

$\Rightarrow \quad x = 2 \text{ or } \frac{18}{5}$

when $x = 2$, $y = \frac{6\times 2}{5\times 2-6}=\frac{12}{4}=3$

when $x = \frac{18}{5}$, $y = \frac{6\times\frac{18}{5}}{5\times\frac{18}{5}-6}=\frac{108}{5\times 12}=\frac{9}{5}$

Hence, $x = 2$, $y = 3$, or $x = 3\frac{3}{5}$, $y = 1\frac{4}{5}$

11. Given equations are:

$$3x - 5y = 2 \text{ and } xy = 8$$

Multiply by x on both the sides of first equation,

$\Rightarrow \quad 3x^2 - 5xy = 2x$

From second equation $xy = 8$

$\Rightarrow \quad 3x^2 - 5 \times 8 - 2x = 0$

$\Rightarrow \quad 3x^2 - 2x - 40 = 0$

$\Rightarrow \quad 3x^2 - 12x + 10x - 40 = 0$

$\quad 3x(x-4) + 10(x-4) = 0$

NOTES

$\Rightarrow \quad (x-4)(3x+10)=0$

$\Rightarrow \quad x=4 \text{ or } -\dfrac{10}{3}$

when $x=4$, $y=\dfrac{8}{x}=\dfrac{8}{4}=2$

when $x=-\dfrac{10}{3}$, $y\dfrac{8}{-10}\times 3=-\dfrac{24}{10}=-\dfrac{12}{5}=-2\dfrac{2}{5}$

Hence, $x=4$, $y=2$;

$x=-3\dfrac{1}{3}, y=-2\dfrac{2}{5}$

12. Given equations are:

$x+y=30$ and $xy=216$

Multiplying both the sides by x of first equation;

$x^2+xy=30x \Rightarrow x^2+216-30x=0$ [since $xy=216$]

$\Rightarrow x^2-30x+216=0$

$\Rightarrow x^2-18x-12x+216=0$

$\Rightarrow x(x-18)-12(x-18)=0$

$\Rightarrow (x-18)(x-12)=0$

$\Rightarrow x=18$ or 12

when $x=18$, $y=\dfrac{216}{18}=12$

when $x=12$, $y=\dfrac{216}{12}=18$

Hence, $x=18$, $y=12$; $x=12$, $y=18$

13. Given equations are:

$x-y=-18$ and $xy=1363$

Multiplying by x in both the sides of first equation,

$x^2-xy=-18x \Rightarrow x^2-1363+18x=0$

$\Rightarrow x^2+18x-1363=0$

$\Rightarrow x^2+47x-29x-1363=0$

$x(x+47)-29(x+47)$

$\Rightarrow (x+47)(x-29)=0$

$x=-47$, or 29 and $y=-29, 47$

Hence, $x=29$, y = 47; $x=-47$, $y=-29$

NOTES

14. Given equations are:

$5x + 2y = 8$ and $9x - 5y = 23$

Multiplying first equation by 5 and second by 2, we get

$25x + 10y = 40$ and

$18x - 10y = 46$

$43x = 86$

$\Rightarrow$ $x = 2$ and from first equation

$5 \times 2 + 2y = 8 \Rightarrow y = -1$

Hence, $x = 2$, $y = -1$

4.9 QUESTIONS AND EXERCISES

Short-Answer Questions

1. What is a pure quadratic equation?
2. What is discriminant?
3. When are roots of a quadratic equation real?
4. Find the value of the discriminant in equation

$$4+\frac{7}{x}+\frac{2}{x^2}=0$$

5. Find the value of C in the given equation so that roots are imaginary

$$3x + 2 + \frac{C}{x} = 0$$

Long-Answer Questions

1. Solve the following equations:

(i) $(3x - 11)(x - 2) + (2x - 3)(x - 4) + 13x = 10(2x - 1)^2 + 12$

(ii) $\frac{x+2}{x-2} - \frac{x-2}{x+2} = \frac{5}{6}$

(iii) $\frac{a + 2x + \sqrt{a^2 - 4x^2}}{a + 2x - \sqrt{a^2 - 4x^2}} = \frac{5x}{a}$

2. Solve the following equations:

(i) $5 - \sqrt{11x^2 - 3x + 1} = 2x$

(ii) $\sqrt{2x+7} + \sqrt{3x-18} = \sqrt{7x+1}$

(iii) $\left(\frac{2x+3}{2x-3}\right)^{1/3} + \left(\frac{2x-3}{2x-3}\right)^{1/3} = \frac{8(4x^2+9)}{13(4x^2-9)}$

NOTES

3. Solve the following equations:

(i) $(x-7)(x-3)(x+5)(x+1) = 1680$

(ii) $(x+9)(x-3)(x-7)(x+5) = 385$

(iii) $16x(x+1)(x+2)(x+3) = 9$

(iv) $(x+2)(x+4)(x+5)(x+10) = 990x^2$

[**Hint.** Rearranging, we get $(x+2)(x+10)(x+4)(x+5) = 990x^2$. Put $x^2 + 20 = t$]

4. Solve the following equations:

(i) $\sqrt{3x^2 - 7x - 30} + \sqrt{2x^2 - 7x - 5} = x + 5$

(ii) $\sqrt{2x^2 + 5x - 2} - \sqrt{2x^2 + 5x - 9} = 1$

(iii) $\sqrt{3x^2 - 2x + 9} + \sqrt{3x^2 - 2x + 4} = 13$

(iv) $\sqrt{x^2 + ax - 1} + \sqrt{x^2 + bx - 1} = \sqrt{a} + \sqrt{b}$

5. Solve the following equations:

(i) $3^{2x} + 9 = 10.3^x$

(ii) $4^x - 3.2^{x+3} = -128$

(iii) $\sqrt{3^x} + \dfrac{3}{\sqrt{3^x}} = 4$

6. Solve the following equations:

(i) $5(5^x + 5^{-x}) = 26$

(ii) $10x^4 - 63x^3 + 52x^2 + 63x + 10 = 0$

(iii) $x^4 - x^3 + 5/4x^2 - x + 1 = 0$

(iv) $4x^4 - 16x^3 + 7x^2 + 16x + 4 = 0$

7. If α, β are the roots of $ax^2 + bx + c = 0$ find the value of

(i) $\dfrac{\alpha}{\beta} + \dfrac{\beta}{\alpha}$

(ii) $\alpha^4 + \beta^4$

(iii) $(\alpha^2 - \beta)^2 + (\beta^2 - \alpha)^2$

(iv) $\alpha^4\beta^7 + \alpha^7\beta^4$

(v) $\left(\dfrac{\alpha}{\beta} - \dfrac{\beta}{\alpha}\right)^2$

NOTES

8. Find the condition that one root of $ax^2 + bx + c = 0$ shall be n times the other.
9. If r is the ratio of the roots of the equation
$$ax^2 + bx + c = 0 \text{ show that } (r + 1)^2\, ac = b^2 r$$
10. Show that the roots of $x^2 - 2ax + a^2 - b^2 - c^2 = 0$ are always real.
11. Prove that the roots of $(a + c - b)x^2 + 2cx + (b + c - a) = 0$ are rational [a, b, c are rational numbers].
12. Solve the following equations:
    (i) $ax + by = 2$, $abxy = 1$
    (ii) $3x - 2y = 7$, $xy = 20$
    (iii) $x^3 + y^3 = 637$, $x + y = 13$
    (iv) $x^3 - y^3 = 218$, $x - y = 2$
13. A horse and a cow were sold for Rs 3040 marking a profit of 25 per cent on the horse and 10 per cent on the cow. By selling them for Rs 3070, the profit realized would have been 10 per cent on the horse and 25 per cent on the cow. Find the cost price of each.
14. Demand for goods of an industry is given by the equation $pq = 100$, where p is the price and q is quantity; supply is given by the equation $20 + 3p = q$. What is the equilibrium price and quantity?
15. Demand and supply equations are $2p^2 + q^2 = 11$ and $p + 2q = 7$. Find the equilibrium price and quantity, where p stands for price and q for quantity.
16. In a perfect competition, the demand curve of a commodity is $D = 20 - 3p - p^2$, and the supply curve is $S = p - 1$, where p is price, D is demand and S is supply. Find the equilibrium price and the quantity exchanged.
17. A man's income from interest and wages is Rs 500. He doubles his investment and also gets an increase of 50 per cent in wages and his income increases to Rs 800. What was his original income separately in terms of interest (I) and wages (W).
18. Solve the following equations:
    (i) $2x + y - 2z = 0$, $7x + 6y - 9z = 0$, $x^3 + y^3 + z^3 = 1728$.
    (ii) $3x + y - 5z = 0$, $7x - 3y - 9z = 0$, $x^2 + 2y^2 + 3z^2 = 23$.
    (iii) $9x + y - 8z = 0$, $4x - 8y + 7z = 0$, $xy + yz + zx = 47$.
19. Solve the following equations:
    (i) $x + y + z = 6$, $x^2 + y^2 + z^2 = 14$, and $x + \frac{y}{2} + \frac{z}{3} = 3$.
    (ii) $\frac{1}{x} + \frac{1}{y} + \frac{1}{z} = 9$, $\frac{2}{x} + \frac{3}{y} = 13$, $8x + 3y = 5$
    (iii) $x(y + z) = 5$, $y(z + x) = 8$, $z(x + y) = 9$.

4.10 FURTHER READING

Khanna, V.K, S.K. Bhambri, C.B. Gupta and Vijay Gupta. *Quantitative Techniques.* New Delhi: Vikas Publishing House.

Khanna, V.K, S.K. Bhambri and Quazi Zameeruddin. *Business Mathematics.* New Delhi: Vikas Publishing House.

UNIT 5 COMPLEX NUMBERS

NOTES

Structure

5.0 INTRODUCTION

We all are familiar with real numbers and use it in our number system. There is also another important class of numbers termed as complex numbers, which consists of imaginary as well as real numbers. In spite of the term imaginary, these numbers are exceptionally used in various areas of science, such as electromagnetic theory, relativity and quantum mechanics.

Girolamo Cardano was an Italian mathematician who first realized the need of finding complex numbers. While solving cubic equations he came across expressions that contained square root of a negative number. Thus, the concept of an imaginary number took birth but this was accepted as a part of mathematical concept by the work of Abraham De Moivre, Bernoullis and Leonhard Euler, who carried out studies on this topic in detail. De Moivre was a French-English mathematician, Bernoullis were mathematicians in a Swiss family and Euler was also a Swiss mathematician. This term 'imaginary' was used by Rene Descartes in 17th century.

NOTES

In this unit, you will learn about complex numbers, their history, their geometrical and graphical representations and performing calculations in complex arithmetic. This unit also discusses quadratic functions, polynomial functions, rational functions with their graphs, matrix representation of complex numbers and important properties of complex numbers.

5.1 UNIT OBJECTIVES

After going through this unit, you will be able to:

- Understand complex numbers
- Describe the Argand plane
- Perform operations using complex numbers
- Explain quadratic functions, polynomial functions, rational function and their graphs
- Discuss the properties of complex numbers

5.2 IMAGINARY NUMBERS

You will now learn about imaginary numbers. For example, take the quadratic equation $x^2 + 1 = 0$. It can also be written as $x^2 = -1$. This equation has no real solutions as every number becomes non-negative when squared. But if we define that the square of imaginary number i is -1, then it is possible to find solution of $x^2 + 1 = 0$. In mathematics, one can describe the notions if they satisfy a specific set of logically constant axioms. For example, the defined number i can also be written as $\sqrt{-1}$, where the real numbers can be included to extend it along with imaginary numbers. The simple method to perform is to describe the set of complex numbers as the set of all numbers of form $z = x + yi$, where x and y are arbitrary real numbers. Here, number x is termed as the real part and y as the imaginary part of the complex number z.

5.2.1 Definition of Complex Number

Complex numbers can be represented in the form of fine geometric interpretation. The real numbers can be viewed as a line on the real number line, similarly the complex numbers can be viewed as a plane termed as the complex plane. Figure 5.1 shows how the complex number $z = x + yi$ with the point (x, y) in the complex plane can be represented:

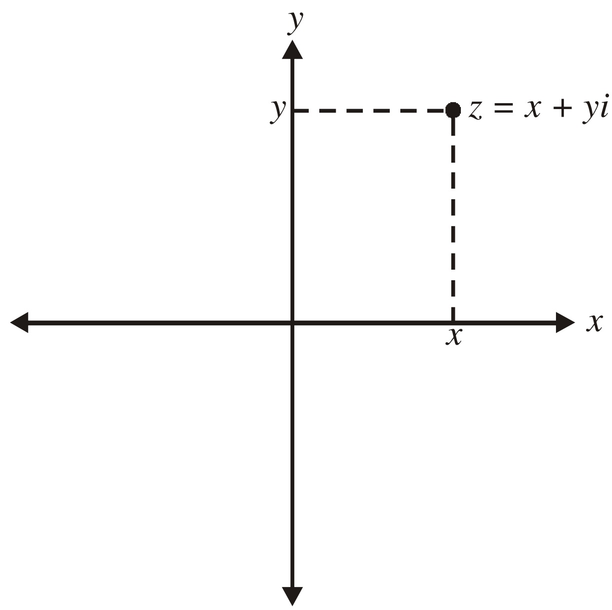


Figure 5.1 Representation of Complex Number z = x + yi with the Point (x, y) in the Complex Plane

NOTES

Basically, there are two functional operations on complex numbers. First is termed as *complex conjugation* and the second is the *modulus*, also famous as the *complex norm* or *absolute value*. For a given complex number $z = x + yi$, one can describe the *complex conjugate* of z (written as z) as the complex number $z = x - yi$ such that in the complex plane, the point corresponding to z is $(x, -y)$. Geometrically, this represents the point (x, y) corresponding to z about the y-axis as shown in Figure 5.2.

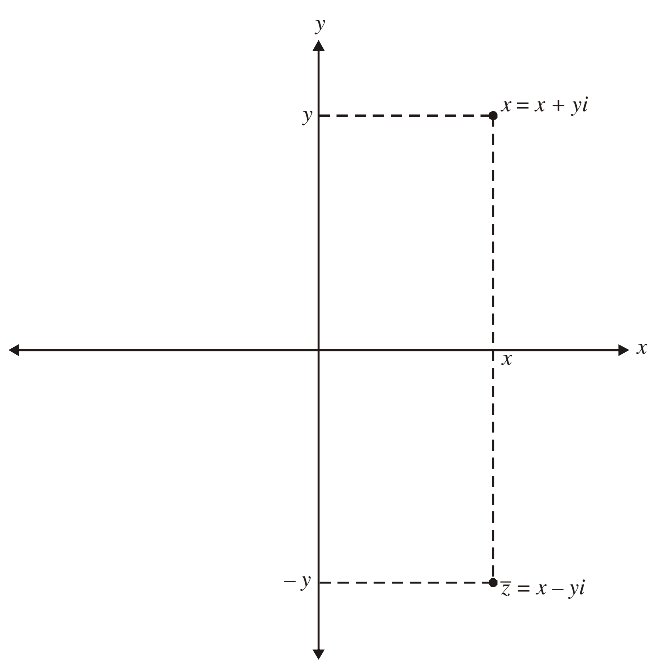


***Figure 5.2** Y-axis*

In case of *modulus*, a given complex number $z = x + yi$ denotes its modulus as $|z|$ and is given by the following formula:

$$|z| = \sqrt{x^2 + y^2}$$

NOTES

As per the distance formula it is equal to the distance from the origin O = (0, 0) up to the point P = (x, y). Hence, the complex number is considered as the length of the line segment passing through the origin to the point in a complex plane.

Rule for Zero: A complex number is equal to zero only when its real part and its imaginary part are equal to zero. For example, $a + bi = 0$ if and only if $a = 0$ and $b = 0$. Similarly, the complex numbers are equal if their real and imaginary parts are equal.

CHECK YOUR PROGRESS

1. How can the quadratic equation $x^2 + 1 = 0$ also be written as?
2. What is the rule for zero?

5.3 COMPLEX NUMBERS: BASIC CHARACTERISTICS

A complex number has two parts, real and imaginary. A number is imaginary when it is the square root of a negative real number. For example, if $x^2 = 1$, then $x = \pm\sqrt{1} = \pm 1$. Such a square root has both real values. But if there is an equation $x^2 + 1 = 0 \rightarrow x^2 = -1$ and hence, $x = \sqrt{-1}$ and this number is imaginary since there is no real number that satisfies this condition then this number, given by $\sqrt{-1}$ is an imaginary number and it is designated by using letter *i*.

A number with an expression containing two terms is known as binomial. Thus, a complex number has two terms in which one is real and the other is imaginary and is multiple of *i*. A complex number is symbolically expressed in the form of a + bi or $x + yi$, where a, b, x and y are real numbers, and i is an imaginary number given as $i = \sqrt{-1}$. This means that $i^2 = -1$. Following are the complex numbers:

$$4 + i,\ 2 - 5\text{i},\ 0.5 + 3i \text{ and } -7 - 3i$$

In a complex number, these two parts cannot be added or subtracted the way it is done in case when all are real numbers.

The powers of i have repetition in a cycle:

$$\begin{aligned}&\ldots\\ &i^{-3} = i\\ &i^{-2} = -1\\ &i^{-1} = -i\\ &i^{0} = 1\end{aligned}$$

NOTES

$i^1 = i$
$i^2 = -1$
$i^3 = -i$
$i^4 = 1$
$i^5 = i$
$i^6 = -1$
...

The same thing can be put in a generalized pattern for any integer n.

$$i^{4n} = 1$$
$$i^{4n+1} = i$$
$$i^{4n+2} = -1$$
$$i^{4n+3} = -i.$$

Observing the above, it can be concluded as: $i^n = i^{n \bmod 4}$.

Example 5.1: Find the Value of i^{83}.

Solution: We divide 83 by 4 and note the remainder; $83 = 20 \times 4 + 3$. Hence, remainder is 3. Thus, $i^{83} = i^3 = i^2.i = -i$

Example 5.2: Find the value of $(i^{72})^2$

Solution: $(i^{72})^2 = i^{144} = i^{4\times36} = (i)^{36} = 1.$

5.3.1 Geometric Representation of Complex Numbers

Real numbers are represented by points in a numerical line as shown in Figure 5.3.

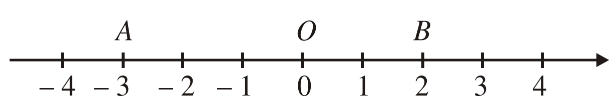


Figure 5.3 *Numerical Line*

In Figure 5.3, point A refers to a number –3 which is on the left side of a number line and hence a negative number. Point B refers to number 2 and O refers to number 0 (zero). On the contrary the complex numbers are represented by points in a numerical coordinate plane. To represent this, select a rectangular Cartesian coordinate with equal scale on both the axes. The complex number $a+ bi$ is represented by point P with abscissa a and ordinate b as shown in Figure5.4. This coordinate system is termed as a *complex plane*.

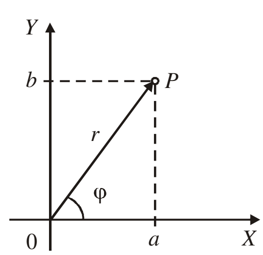


Figure 5.4 *Coordinate System*

NOTES

Modulus of a complex number is a length of vector OP which represents the complex number in a coordinate *complex* plane. Modulus of complex number $a+bi$ is signed as

$|a+bi|$ or by letter r and is equal to :

$$r = |a+bi| = \sqrt{a^2+b^2}.$$

Conjugate complex numbers have the same modulus.

Argument of a complex number is the angle φ between x-axis and vector OP, representing this complex number. Hence, $\mathbf{\tan \varphi = b/a}$.

Trigonometric form of a complex number. Abscissa a and ordinate b of the complex number $a + bi$ can be expressed by its modulus r and argument φ.

Then, $a = r \cos \varphi$, $b = r \sin \varphi$

$a + bi = r(\cos \varphi + i \sin \varphi)$

Operations with Complex Numbers Represented in the Trigonometric Form

1. $z_1 \cdot z_2 = [r_1(\cos \varphi_1 + i \sin \varphi_1)]\,[r_2(\cos\varphi_2 + i \sin \varphi_2)] =$
   $= r_1 \cdot r_2\,[\cos(\varphi_1 + \varphi_2) + i \sin(\varphi_1 + \varphi_2)]$
2. $z_1/z_2 = [r_1(\cos \varphi_1 + i \sin \varphi_1)]/[r_2(\cos \varphi_2 + i \sin \varphi_2)] =$
   $= r_1/r_2\,[\cos(\varphi_1 - \varphi_2) + i \sin(\varphi_1 - \varphi_2)]$
3. $z^n = [r(\cos \varphi + i \sin \varphi)]^n = r^n(\cos n\varphi + i \sin n\varphi)$

This is the famous *Moivre's formula*.

4. $\sqrt[n]{z} = \sqrt[n]{r(\cos\varphi + i\sin\varphi)} = \sqrt[n]{z}\,\{\cos[(\varphi + 2\pi k)/n] + i\sin[(\varphi + 2\pi k)/n]\}$

Here k is any integer. To receive n different values of the nth degree root of z, it is essential to give n consecutive values for k, for example $k = 0, 1, 2, \ldots, n - 1$.

5.3.2 Complex Arithmetic

Addition and subtraction can be performed on complex numbers easily. To do this, simply add or subtract the respective real and imaginary parts of each complex number. For example, the sum of two arbitrary complex numbers $z_1 = x_1 + y_1 i$ and $z_2 = x_2 + y_2 i$ is as follows: $z_1 + z_2 = (x_1 + x_2) + (y_1 + y_2)\, i$ and their difference is given by $z_1 - z_2 = (x_1 - x_2) + (y_1 - y_2)\, i$.

Multiplication and division of complex numbers are very complex. To describe complex multiplication the distributive property is used as follows:

$$(a + b)\, c = ac + bc \text{ and } a\,(b + c) = ab + ac$$

When both these rules are applied we derive the additional property of foil rule as follows:

$$(a + b)(c + d) = ac + ad + bc + bd$$

NOTES

Apply the foil rule for computing the product of the above given complex numbers z_1 and z_2. If $a = x_1$, $b = y_1\, i$, $c = x_2$, and $d = y_2\, i$, then $z_1\, z_2 = (x_1 + y_1\, i)(x_2 + y_2\, i) = x_1\, x_2 + x_1\, y_2\, i + y_1\, x_2\, i + y_1\, y_2\, i^2$. *If* $i^2 = -1$, then $z_1\, z_2 = (x_1\, x_2 - y_1\, y_2) + (x_1\, y_2 + y_1\, x_2)\, i$.

Using this formula any two complex numbers can be multiplied. The following identity is verified using the above derived complex number definition:

$$|z| = \sqrt{z\bar{z}}.$$

Hence, the complex number is equal to the square root of the product of the number and its complex conjugate. This definition is also used to describe complex division. Division on complex numbers is very complex. For example, let the quotients are z_1/z_2, where z_1 and z_2 are arbitrary complex numbers and z_2 is nonzero. When the numerator and denominator is multiplied by the complex conjugate of z_2, then the result is $(z_1/z_2)(\bar{z}_2/\bar{z}_2) = z_1\bar{z}_2 / |z_2|^2$.

To divide two complex numbers in terms of the real and imaginary parts of z_1 and z_2 is shown below:

$$\frac{z_1}{z_2} = \frac{x_1 + y_1 i}{x_2 + y_2 i} = \frac{(x_1 x_2 + y_1 y_2) + (x_2 y_1 - x_1 y_2)i}{x_2^2 + y_2^2}$$

Operations on complex numbers are like those on real numbers in many ways. But there are two exceptions of these rules. Two exceptions to this are:

1. Addition of subtraction of the two terms of a complex number ***a* + *bi*** can not be done and should be left as it is.
2. An expression $i^2 = -1$, is against the general rule that product of two numbers having same sign is positive. If we take $i.i = i^2 = -1$. If we take it like $i.i = \sqrt{-1}.\sqrt{-1} = \sqrt{(-1)(-1)} = \sqrt{1} = 1$. Two results are contradictory.

5.3.3 Operations on Complex Numbers

The following are the general rules for operations on complex numbers:

1. **Equality:** Two complex numbers are equal if their real parts and imaginary parts are each equal. If two complex numbers c1 = $x + yi$ and c2 = $w + zi$ and c1 = c2, this means $x + yi = w + zi$ and this leads to the fact that $x = w$ and $y = z$.
2. **Addition:** If there are two complex numbers $a + bi$ and $c + di$ and if they are added together, their real parts are added together and the same applies to their imaginary parts too. For example,

$$(a + bi) + (c + di) = (a + c) + (b + d)i$$

Here, the real part of first is added to the real part of the other and the imaginary part of one is added to the imaginary part of the other.

Example 5.3: If $z_1 = (2 + 3i)$ and $z_2 = (5 - 9i)$, find $z_1 + z_2$.

Solution: $z_1 + z_2 = (2 + 5) + (3 + (-9))i = 7 - 6i$.

NOTES

3. **Subtraction:** If one complex number $c + di$, is subtracted from another complex number $a + bi$, then real part c, is subtracted from real part a, and imaginary part d, is subtracted from the imaginary part b. This is shown as below:

   $(a + bi) - (c + di) = (a - c) + (b - d)i$

Example 5.4: If If $z_1 = (3 + 11i)$ and $z_2 = (7 + 13i)$, find $z_1 - z_2$.

Solution: $z_1 - z_2 = (3 + 11i) - (7 + 13i) = -4 - 2i$

4. **Zero:** If a complex number is zero, then both the parts are separately zero. For example, if $x + yi = 0$, this means that $x = 0$ and $y = 0$.
5. **Opposites:** To find opposite of a complex number, change the sign of each part. If $a + bi$ is a complex number, its opposite id found by its negation, i.e., as $-(a + bi)$ that leads to $-a + (-b)i = -a - bi$. Opposite of a complex number $7 - 4i$ is $-7 + 4i$.
6. **Multiplication:** Product of two complex numbers is also a complex number. If z_1 and z_2 are two complex numbers where $z_1 = x_1 + y_1 i$ and $z_2 = x_2 + y_2 i$, their product $z_1 \times z_2$ is given by $(x_1 + y_1 i) \times (x_2 + y_2 i) = x_1x_2 + x_1y_2 i + y_1x_2 i + y_1y_2 i^2$. Since $i^2 = -1$, $y_1y_2 i^2 = -y_1y_2$.

   Thus, $z_1 \times z_2 = (x_1x_2 - y_1y_2) + (x_1y_2 + y_1x_2)i$ and this too is a complex number having $(x_1x_2 - y_1y_2)$ as its real part and $(x_1y_2 + y_1x_2)$ as its imaginary part.
7. **Conjugates:** Every complex number has conjugate that is found just by changing the sign of its imaginary part. If $x + yi$ is a complex number its conjugate is $x - yi$. Thus, $x + yi$ and $x - yi$ are conjugate pairs known as complex conjugates. This is denoted by placing a bar over the symbol. If $\boldsymbol{z = x + yi}$, is a complex number, its conjugate is denoted as $\overline{z} = x - yi$ and $z \times \overline{z}$ is given by $x^2 + y^2$. Thus, product of a conjugate pair is always a real number.

Example 5.5: A complex number is given as $z = 7 + 5i$. Find its conjugate and also find the product of the conjugate pair.

Solution: If $z = 7 + 5i$ then its conjugate is given by $\overline{z} = 7 - 5i$. Their product $z \times \overline{z}$ is given by $7^2 + 5^2 = 49 + 25 = 77$.

8. **Division:** If a complex number is divided by another complex number, it gives a complex number. Thus, the result of division of one complex number by another is also a complex number. If $z_1 = x_1 + y_1 i$ and $z_2 = x_2 + y_2 i$ are two complex numbers and z_1 is divided by z_{2+} then, z_2 can not be zero, since division by zero is not allowed. Also, complex number in the denominator side should be converted to a real number to perform this

division. To do this multiplication is done by the complex conjugate of denominator should be multiplied to both numerator as well as denominator.

If we are required to find z_1/z_2 we must first multiply conjugate of z_2 to numerator as well as denominator. One example will make it clear.

Example 5.6: Two complex numbers are given as; $z_1 = 5 + 15i$ and $z_2 = 4 + 3i$. Find z_1/z_2.

Solution: We have to find $z_1/z_2 = (5 + 15i)/(4 + 3i)$. Complex conjugate of z_2 is $4 - 3i$. We multiply this to both, numerator as well as denominator and we get $z_1/z_2 = (5 + 15i)(4 - 3i)/(4 + 3i)\,(4 - 3i) = (20 - 15i + 60i - 45i^2)/(4^2 - 3^2i^2)$

$\Rightarrow z_1/z_2 = (20 + 45 + 45i)/(16 + 9)$

and hence, $z_1/z_2 = \dfrac{65+45i}{25} = \dfrac{13}{5} + \dfrac{9}{5}i$

Check Your Progress

3. What are the two parts of a complex number?
4. Which are those arithmetic operations that can be performed easily in complex numbers and which are the ones that are complex?

5.3.4 Graphical Representation

Two parts of a complex number can be represented graphically by representing one along the real axis (shown horizontally) and another imaginary axis, at positive right angle to it. This is like presenting a point on a Cartesian plane by an ordered pair (x, y). Thus, a complex number $x + yi$ can be denoted as a coordinate pair (x, y), as shown in Figure 5.5.

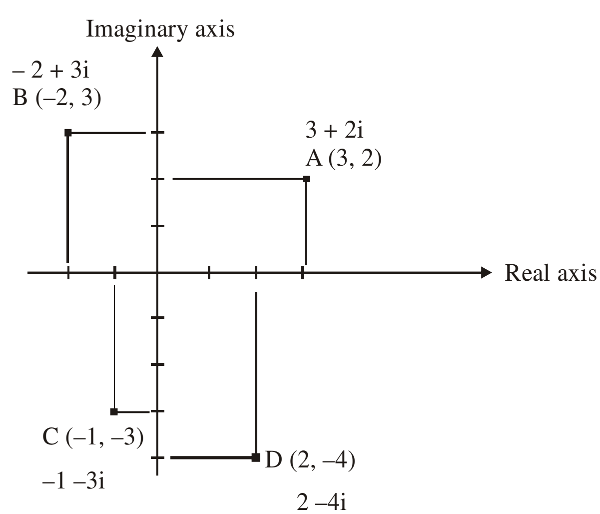


Figure 5.5 *Graphical Representation of Complex Number*

NOTES

Argand Diagram

An Argand plane is a way to represent a complex number as points on rectangular coordinate plane. This also known as complex plane in which x-axis is used for real axis and imaginary axis is the y-axis. Argand plane is so named as an amateur mathematician named Jean Robert Argand described the plane in his paper in the year 1806. Some 120 years earlier a similar method was suggested by John Wallis. His work was developed by Casper Wessel who published his paper in Danish that was not the common language used for mathematics during that period. After 1895 his work was noticed, but by that time the name 'Argand diagram' was known by a community of mathematicians. Figure 5.6 shows the position of a complex number on Argand plane.

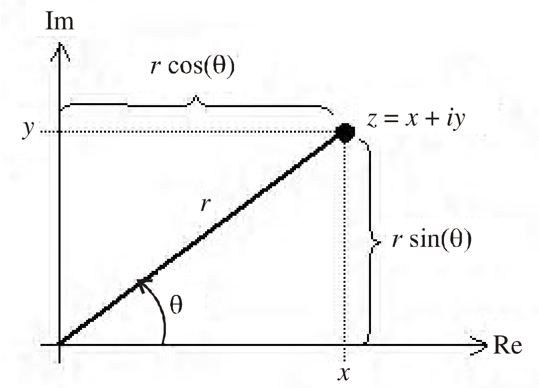


Figure 5.6 *Position of a Complex Number on an Argand Plane*

On an Argand plane, the position of a point can be shown both in rectangular coordinate system as well as polar coordinate system. A complex number z has been shown as an ordered pair (x, y) on a Cartesian plane and as (r, θ) in polar coordinates.

The point shown by z has length r and makes a positive angle of q from the horizontal axis. Here, $x = r\cos\theta$ and $y = r\sin\theta$. The angle θ is given as $\theta = \tan^{-1}(y/x)$ since $\tan\theta = y/x$.

The distance of this point from the origin is given as $r^2 = x^2 + y^2$. Here, r is known as the modulus of the complex number z and is written as $|z|$. Thus, $r = |z| = \sqrt{(x^2 + y^2)}$. Now the entire thing can be presented in brief with the help of Figure 5.7.

NOTES

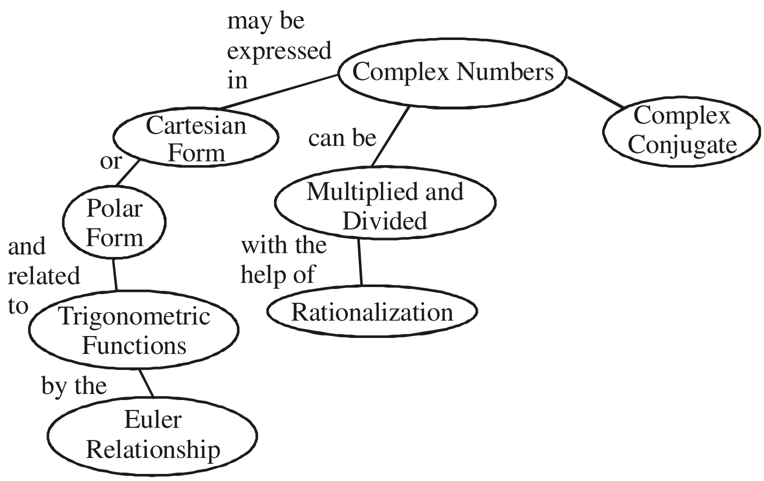


Figure 5.7 *An Argand Plane*

Thus, a complex number of the form $z = x + yi$ or $z = x + iy$ can also be written as $z = r(\cos\theta + i\sin\theta)$ or $z = |z|(\cos\theta + i\sin\theta)$ where, $x = r\cos\theta$, $y = r\sin\theta$ and $\theta = \tan^{-1}(y/x)$. This formulae forms a link between algebraic and trigonometric quantities. Here, θ is also known as argument of the complex number z and written as $\theta = \text{Arg}(z)$ and r is modulus of z and also known as absolute value.

Leonhard Euler in the year 1748 found a formula that in known as Euler's formula in his name that was used for complex analysis. This is given as below:

$$\cos\theta + i\sin\theta = e^{i\theta}.$$

5.3.5 Quadratic Functions and their Graphs

The quadratic equations are of the form $ax^2 + bx + c = 0$, where a, b and c are considered as real-valued constants. The left part of this equation is a quadratic function of the form $f(x) = ax^2 + bx + c$ and the graph of this quadratic function is termed as parabola, which is a special type of curve. The simple parabola is specified by the function $f(x) = x^2$. Take the quadratic function $f(x) = ax^2 + bx + c$. The roots of this specific function are the results of the equation $f(x) = 0$ which are given by the quadratic formula as,

$$x = (-b \pm \sqrt{b^2 - 4ac}) / 2a$$

The quantity $D = b^2 - 4ac$ is termed as *discriminant* of $f(x)$ within the radical. The sign of the discriminant, D, is used to determine the number of real roots of f. When D is positive then f will have two distinct real roots, when D is zero then f will have just one real root, and when D is negative then f will have no real roots. First compute the roots of a quadratic equation before representing it graphically.

Example 5.7: Graphically represent the quadratic function $f(x) = x^2 + x - 2$.

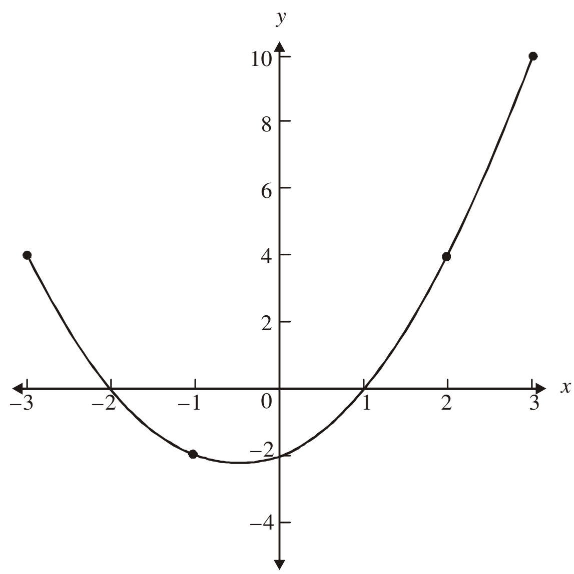
Solution: By factorizing, we get the factors of $f(x)$ as $(x - 1)(x + 2)$. Hence, the roots of f are 1 and –2. Now compute the values of $f(x)$ for values of x close to these roots. The following table sums up the results.

Results of Quadratic Function $f(x) = x^2 + x - 2$.

NOTES

x	$f(x)$
–3	4
–2	0
–1	–2
0	–2
1	0
2	4
3	10

Plot a smooth curve to connect these points. The following graph will be obtained.



Results of the Quadratic Function $f(x) = x^2 + x - 2$

Example 5.8: Graphically represent the quadratic function $f(x) = 2 + 2x - x^2$.

Solution: For finding the roots of f apply the quadratic formula $x = (-b \pm \sqrt{b^2 - 4ac}) / 2a$, where $a = -1$, $b = 2$ and $c = 2$ to obtain the following result:

$$x = (-2 \pm \sqrt{4 - (4)(-1)(2)}) / (2)(-1)$$
$$= (-2 \pm \sqrt{12})/ -2$$
$$= 1 \pm \sqrt{3}$$

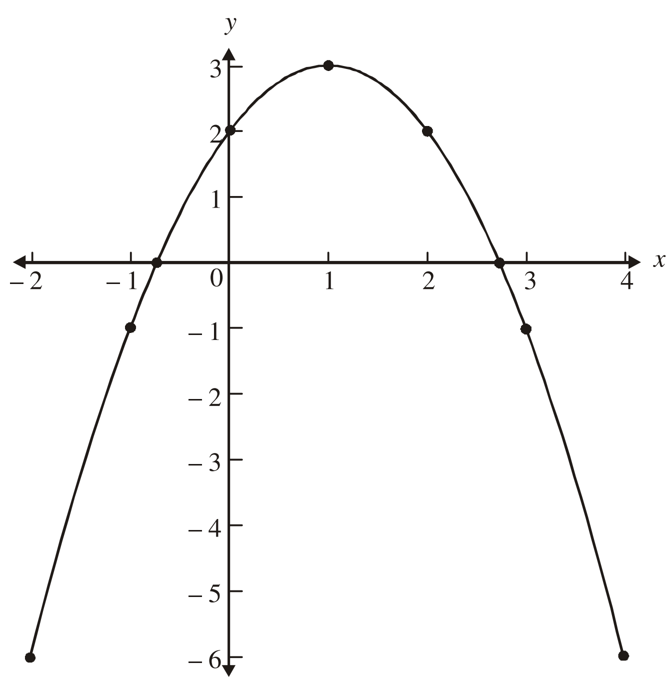
Hence, the roots of f are $1 - \sqrt{3} \approx -0.73$ and $1 + \sqrt{3} \approx 2.73$. Compute again the values of $f(x)$ for values of x close to these roots. The results are tabulated as follows:

Results of the Quadratic Function $f(x) = 2 + 2x - x^2$

x	$f(x)$
–2	–6
–1	–1
–0.73	0
0	2
1	3
2	2
2.73	0
3	–1
4	–6

NOTES

Finally, plot these points and connect them by drawing a smooth curve. The following graph will be obtained.



Graphical Representation of the Quadratic Function $f(x) = 2 + 2x - x^2$

5.3.6 Polynomial Functions and their Graphs

This section explains the methodology to graph common polynomial functions. The complexity of the graph depends on the degrees of polynomials growth. A polynomial of degree n will have up to n real zeros so that its graph crosses the x–axis n times. However, one can easily find the zeros of some polynomials by simply plotting sufficient points. Reasonably exact graphs can be made for general polynomials.

NOTES

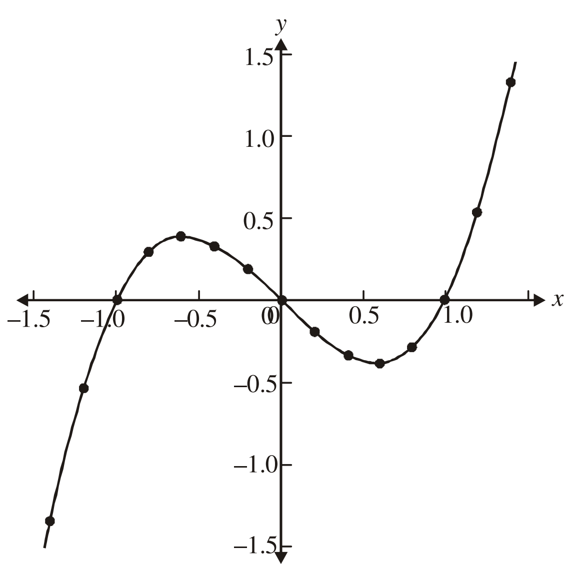
Example 5.9: Graphically represent the polynomial $f(x) = x^3 - x$.

Solution: Factorize $f(x)$. Consider that $f(x) = x\,(x^2 - 1)$. According to rule if $a = 1$ then the factors of $x^2 - 1$ will be $(x + 1)\,(x - 1)$. Hence, $f(x) = x(x + 1)(x - 1)$. The zeros of f will be at -1, 0 and 1. Computing the values of $f(x)$ for various close points, you will obtain the following table.

Result of the Quadratic Function $f(x) = x^3 - x$

x	$f(x)$
–1.4	–1.344
–1.2	–0.528
–1.0	0.000
–0.8	0.288
–0.6	0.384
–0.4	0.336
–0.2	0.192
0.0	0.000
0.2	–0.192
0.4	–0.336
0.6	–0.384
0.8	–0.288
1.0	0.000
1.2	0.528
1.4	1.344

Plot these points and connect them with a smooth curve to obtain the following graph.



Graphical Representation of the Quadratic Function $f(x) = x^3 - x$

***Note*:** This graph is symmetric with reference to the origin because f is odd.

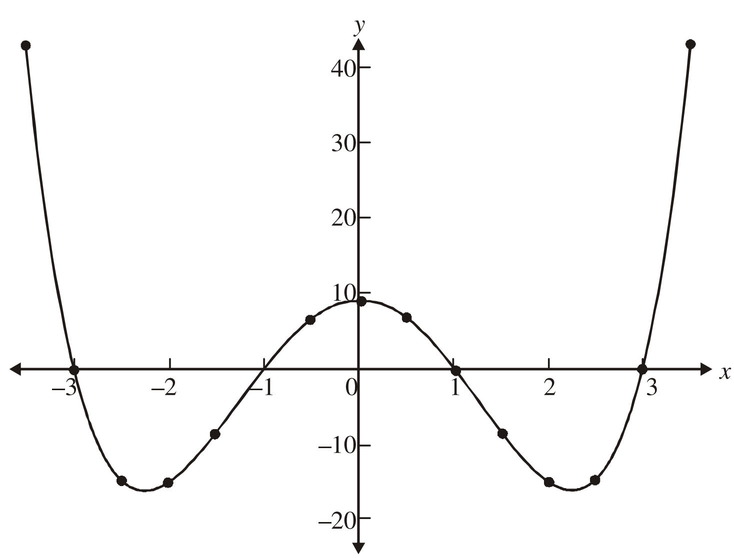
Example 5.10: Graphically represent the polynomial f(x) = $x^4 - 10x^2 + 9$.

Solution: After factorizing this function we have $f(x) = (x^2 - 1)(x^2 - 9) = (x + 1)(x - 1)(x + 3)(x - 3)$. Hence, f will have zeros at ±1 and ±3. The values of $f(x)$ where x covers the range of zeros are listed in the following table. Here, values of $f(x)$ are rounded to the nearest tenth.

Results of the Polynomial $f(x) = x^4 - 10x^2 + 9$

x	$f(x)$
–3.5	36.6
–3.0	0.0
–2.5	–14.4
–2.0	–15.0
–1.5	–8.4
–1.0	0.0
–0.5	6.6
0.0	9.0
0.5	6.6
1.0	0.0
1.5	–8.4
2.0	–15.0
2.5	–14.4
3.0	0.0
3.5	36.6

Plot these point and connect them to get the following graph.



Graph of f(x)

Note: This graph is symmetric with reference to the y-axis because f is even.

Example 5.11: Graphically represent the polynomial $f(x) = x^3 + x^2 + 2x + 4$.

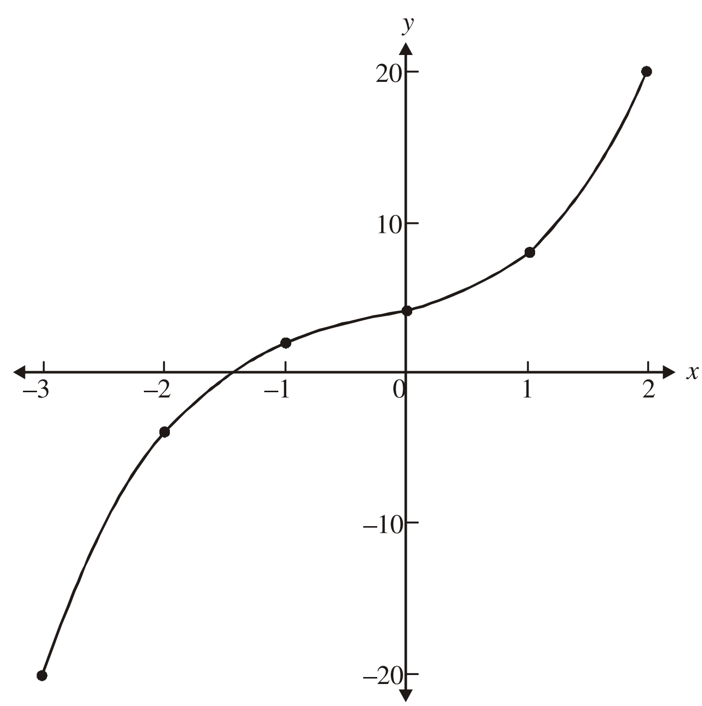
Solution: It is difficult to factorize this polynomial. Hence, $f(x)$ is computed for certain small values of the argument.

NOTES

Result of the polynomial $f(x) = x^3 + x^2 + 2x + 4$

x	$f(x)$
–3	–20
–2	–4
–1	2
0	4
1	8
2	20

The following graph is obtained after plotting the values.



Graphical Representation of the Polynomial $f(x) = x^3 + x^2 + 2x + 4$

Note: Here, the function $f(x)$ is neither even nor odd and so it has only one real zero.

NOTES

5.3.7 Division of Univariate Polynomials

The following theorems are used for dividing univariate polynomials.

Theorem: If $p_1(x)$ and $p_2(x)$ are univariate polynomials, then there exist unique polynomials $q(x)$ and $r(x)$ such that $p_1(x) = q(x)\, p_2(x) + r(x)$ with $\deg(r) < \deg(p_2)$.

In this expression, the polynomial $q(x)$ is termed as the *quotient* of the polynomials $p_1(x)$ and $p_2(x)$ and the polynomial $r(x)$ is termed as the *remainder* of $p_1(x)$ and $p_2(x)$. For finding $q(x)$ and $r(x)$ long polynomial division method is used.

Example 5.12: Given is polynomial $p_1(x) = 8x^4 + 16x^3 - 10x^2 + 21x - 25$. Find the quotient and remainder of this polynomial divided by the polynomial $p_2(x) = 2x^2 + 3x - 1$.

Solution: The polynomial division is done as follows:

$$
\begin{array}{r|rrrrr}
 & 4x^2 & +2x & -6 & & \\
\hline
2x + 3x - 1 & 8x^4 & 16x^3 & -10x^2 & +21x & -25 \\
 & -8x^4 & -12x^3 & +4x^2 & & \\
\cline{2-4}
 & & 4x^3 & -6x^2 & +21x & \\
 & & -4x^3 & -6x^2 & +2x & \\
\cline{3-6}
 & & & -12x^2 & +23x & -25 \\
 & & & 12x^2 & +18x & -6 \\
\cline{4-6}
 & & & & 41x & -31
\end{array}
$$

Hence, we get $q(x) = 4x^2 + 2x - 6$ and $r(x) = 41x - 31$, where $8x^4 + 16x^3 - 10x^2 + 21x - 25 = (4x^2 + 2x - 6)(2x^2 + 3x - 1) + 41x - 31$. Its equivalent is,

$$\frac{8x^4+16x^3-10x^2+21x-25}{2x^2+3x-1} = 4x^2+2x-6+\frac{41x-31}{2x^2+3x-1}$$

There is a fine technique termed as *synthetic division* which is used to divide a polynomial by a linear polynomial.

Example 5.13: Find the quotient and remainder of the polynomial $p_1(x) = x^3 + 6x^2 + 11x + 8$ divided by the polynomial $p_2(x) = x + 2$.

Solution: Use the following long polynomial division method:

$$
\begin{array}{r|rrrr}
 & x^2 & +4x & +3 & \\
\hline
x + 2 & x^3 & +16x^2 & +11x & +8 \\
 & -x^3 & -2x^2 & & \\
\cline{2-3}
 & & 4x^2 & +11x & \\
 & & -4x^2 & -8x & \\
\cline{3-5}
 & & & 3x & +8 \\
 & & & -3x & -6 \\
\cline{4-5}
 & & & & 2
\end{array}
$$

Hence, we get $(x^3 + 6x^2 + 11x + 8) / (x + 2) = x^2 + 4x + 3 + 2 / (x + 2)$.

NOTES

Synthetic division gives a short cut method for the same calculation. In synthetic division, we just write the rows of numbers and not the powers of the variable x. In the first/top row, the coefficients of the dividend are written which is preceded to the left by the negative of the constant coefficient of the divisor divided by the linear coefficient. In this case it is 1 and the result is –2. This is the number of root which is being tested. Down two rows, write the leading coefficient of the dividend. In the second row this coefficient is multiplied by the root –2 and the result is written one space towards right. It is shown below. Then the corresponding coefficient is added in the top row that is 6 to the result 4 in the third row. Now multiply this number by the root to obtain –8 which is written in the second row. It is continued till each row is completed up to the rightmost coefficient of the dividend as shown below:

$$\begin{array}{r|rrrr} -2 & 1 & 6 & 11 & 8 \\ & & -2 & -8 & -6 \\ \hline & 1 & 4 & 3 & 2 \end{array}$$

As a result, the quotient is $x^2 + 4x + 3$ and the remainder is 2.

5.3.8 Zeros of Polynomial Functions

Finding zeros of polynomial function is the important key of a function and is termed as roots. These are the values of the variable x for $f(x) = 0$. Principally, it is extremely useful in finding the zeros of a univariate polynomial whereas to find the zeros of linear polynomials is completely trivial.

Theorem: The linear polynomial $x - a$ is a factor of the polynomial $p(x)$ if and only if a is a zero of f and also $p(a) = 0$.

Proof: Evidently if $x - a$ is a factor of $p(x)$ then p(a) should be zero. This factor becomes zero when $x = a$. Consider $p(a) = 0$. As per Theorem, there exists polynomials $q(x)$ and $r(x)$ with $\deg(r) < \deg(x - a) = 1$ such that $p(x) = q(x)$ $(x - a) + r(x)$. Because $\deg(r)$=0, hence, $r(x)$ should be equal to a constant r. As a result we have $p(x) - q(x)\ (x - a) + r$. But we know that $p(a) = q(a)\ (a - a) + r = r = 0$, where $r = 0$ and $p(x) = q(x)\ (x - a)$. Hence, $x - a$ is factor of $p(x)$.

Theorem: Let $p(x) = c_n x^n + c_{n-1} x^{n-1} + ... + c_2 x^2 + c_1 x + c_0$ be a polynomial of degree n with integer coefficients and leading coefficient c_n nonzero. Again, let a be a rational root of $p(x)$ then there exists integers j and k with j dividing c_0 and k dividing c_n such that $a = j / k$.

Proof: According to the above Theorem, a is zero of $p(x)$ if and only if $x - a$ is a linear factor of $p(x)$. Because a is rational, hence a must have the form j / k for some integers j and k with k nonzero. So, $x - a = x - j/k$ is a linear factor of $p(x)$. But this implies that $(k)(x - a) = (k)(x - j/k) = kx - j$ is also a linear factor of $p(x)$. Let $q(x) = p(x)/(kx - j)$. Now $q(x)$ is a polynomial of degree $n - 1$ with integer coefficients, so that we have $q(x) = b_{n-1}x^{n-1} + \ldots + b_2x^2 + b_1x + b_0$. But $p(x) = (b_{n-1}x^{n-1} + \ldots + b_2x^2 + b_1x + b_0)(kx - j) = c_nx^n + c_{n-1}x^{n-1} + \ldots + c_2x^2 + c_1x + c_0$, imply that $kb_{n-1} = c_n$ and $-jb_0 = c_0$. Thus, j divides c_0 and k divides c_n as claimed.

NOTES

Example 5.14: Find the rational zeros of the polynomial $p(x) = x^3 + x^2 - 4x - 4$.

Solution: According to above mentioned Theorem, the possible rational zeros of $p(x)$ are ±1, ±2 and ±4. Each of these zeros is tested by synthetic division till one is found. The following is the result:

$$
\begin{array}{r|rrrr}
1 & 1 & 1 & -4 & -4 \\
 & & 1 & 2 & -2 \\
\hline
 & 1 & 2 & -2 & \vert\, -6
\end{array}
\qquad
\begin{array}{r|rrrr}
-1 & 1 & 1 & -4 & -4 \\
 & & -1 & 0 & 4 \\
\hline
 & 1 & 0 & -4 & \vert\, 0
\end{array}
$$

Notice that 1 is not a zero of $p(x)$ instead -1 is. We also have $p(x) = (x + 1)(x^2 - 4) = (x + 1)(x + 2)(x - 2)$ to get the other two zeros, namely -2 and 2. It is simply obtained by factoring the quotient $x^2 - 4$.

Example 5.15: Find the rational zeros of the polynomial $p(x) = 2x^3 - x^2 - x - 3$.

Solution: As per the Theorem, the possible rational zeros of $p(x)$ are ±1, ±3, $\pm1/2$ and $\pm3/2$. Each of these zeros is tested using synthetic division till one is found. The following is the result:

$$
\begin{array}{r|rrrr}
1 & 2 & -1 & -1 & -3 \\
 & & 2 & 1 & 0 \\
\hline
 & 2 & 1 & 0 & \vert\, -3
\end{array}
\qquad
\begin{array}{r|rrrr}
-1 & 2 & -1 & -1 & -3 \\
 & & -2 & 3 & -2 \\
\hline
 & 2 & -3 & 2 & \vert\, -5
\end{array}
$$

$$
\begin{array}{r|rrrr}
3 & 2 & -1 & -1 & -3 \\
 & & 6 & 15 & 42 \\
\hline
 & 2 & 5 & 14 & \vert\, 39
\end{array}
\qquad
\begin{array}{r|rrrr}
-3 & 2 & -1 & -1 & -3 \\
 & & -6 & 21 & -60 \\
\hline
 & 2 & -7 & 20 & \vert\, -63
\end{array}
$$

$$
\begin{array}{r|rrrr}
\frac{1}{2} & 2 & -1 & -1 & -3 \\
 & & 1 & 0 & -\frac{3}{2} \\
\hline
 & 2 & 0 & -1 & \vert\, -\frac{9}{2}
\end{array}
\qquad
\begin{array}{r|rrrr}
-\frac{1}{2} & 2 & -1 & -1 & -3 \\
 & & -1 & 1 & 0 \\
\hline
 & 2 & -2 & 0 & \vert\, -3
\end{array}
$$

$$
\begin{array}{r|rrrr}
\frac{3}{2} & 2 & -1 & -1 & -3 \\
 & & 3 & 3 & 3 \\
\hline
 & 2 & 2 & 2 & \vert\, 0
\end{array}
\qquad
\begin{array}{r|rrrr}
-\frac{3}{2} & 2 & -1 & -1 & -3 \\
 & & -3 & 6 & -\frac{15}{2} \\
\hline
 & 2 & -4 & 5 & \vert\, -\frac{21}{2}
\end{array}
$$

Notice that 3/2 is the only rational zero of p(x). We also have the factorization $p(x) = (x - 3/2)(2x^2 + 2x + 2) = (2x - 3)(x^2 + x + 1)$. Actually, 3/2 is the only real zero of $p(x)$ as the cofactor $x^2 + x + 1$ does not has real zeros.

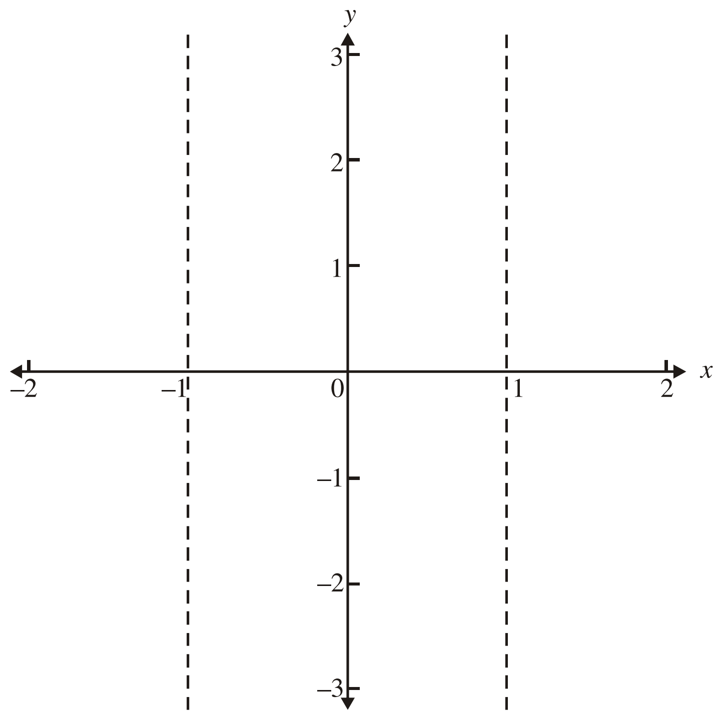
5.3.9 Rational Functions and their Graphs

NOTES

Rational functions have the common form $f(x) = p(x)/q(x)$, where p and q are polynomials. Every time $q(x)$ is zero and $p(x)$ is not, $f(x)$ remains undefined. This results as singularity of $f(x)$. For values of x close to the singular value, $|f(x)|$ becomes very large. Hence, while graphing rational functions first get its singularities.

Example 5.16: Graphically represent the function $f(x) = 1/(1 - x^2)$.

Solution: First compute the singularities of function before you graph f. The denominator is $1 - x^2$, whose factors are $(1 + x)(1 - x)$ and has zeros at $x = \pm 1$. Because the numerator of f is never zero, hence these values must be the singularities of f. Draw vertical dashed lines, termed as *asymptotes* for these values of x, as shown in the following figure.



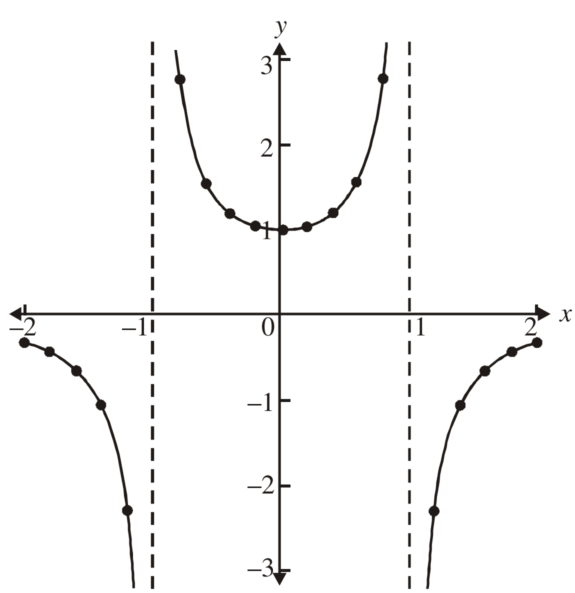
Graphical Representation of the Function $f(x) = 1/(1 - x^2)$

Now prepare a table on values of $f(x)$ vs values of x. Fundamentally, different values of x are used which are close to the singular values. These values are used to construct the table. The following table reproduces the values for constructing a graph along with the asymptotes.

Values for Constructing Graph along with the Asymptotes

x	$f(x)$
−2.0	−0.33
−1.8	−0.45
−1.6	−0.64
−1.4	−1.04
−1.2	−2.27
−1.0	–
−0.8	2.78
−0.6	1.56
−0.4	1.19
−0.2	1.04
0.0	1.00
0.2	1.04
0.4	1.19
0.6	1.56
0.8	2.78
1.0	–
1.2	−2.27
1.4	−1.04
1.6	−0.64
1.8	−0.45
2.0	−0.33

This table is graphically presented as follows:



Graph Showing f(x) vs Values of x

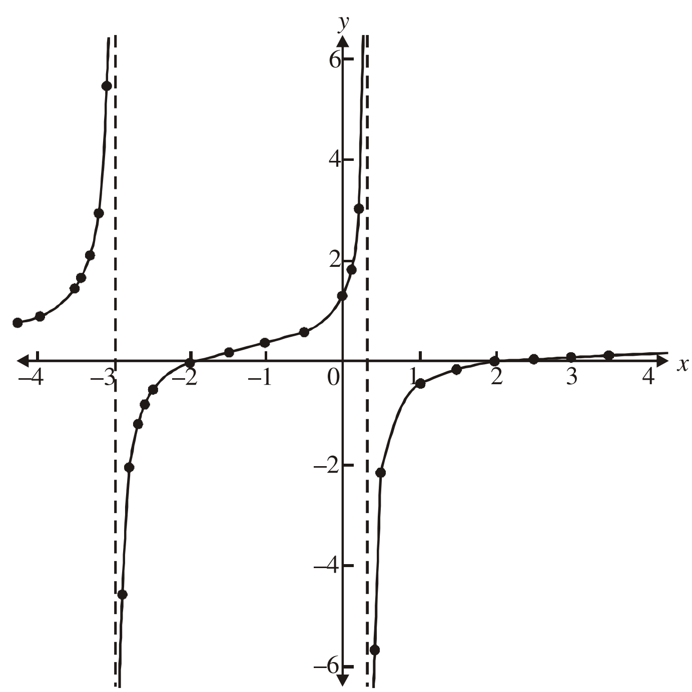
NOTES

Example 5.17: Graphically represent the function $f(x) = (x^2 - 4) / (3x^2 + 8x - 3)$.

Solution: Using the trial and error find the denominator factors of $3x^2 + 8x - 3$ which are $(x + 3)(3x - 1)$ and has zeros at $x = -3$ and $x = 1/3$, whilst the numerator has zeros at $x = \pm 2$. Now compute $f(x)$ for values of x starting at –4 and ending at 4 along with additional values near $x = 1/3$ and $x = 3$. The following table shows these values which are represented in a graph.

Additional Values

x	$f(x)$	x	$f(x)$
−4.0	0.92	−0.5	0.60
−3.5	1.43	0.0	1.33
−3.4	1.69	0.1	1.84
−3.3	2.11	0.2	3.09
−3.2	2.94	0.3	11.85
−3.1	5.45		
−3.0	–	0.4	−5.65
−2.9	−4.55	0.5	−2.14
−2.8	−2.04	1.0	−0.38
−2.7	−1.21	1.5	−0.11
−2.6	−0.78	2.0	0.00
−2.5	−0.53	2.5	0.06
−2.0	0.00	3.0	0.10
−1.5	0.21	3.5	0.13
−1.0	0.38	4.0	0.16



Graphical Representation of the Function $f(x) = (x^2 - 4/3x^2 + 8x - 3)$

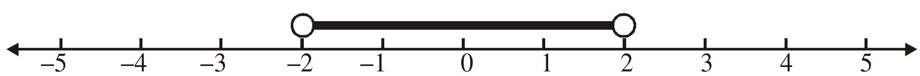
NOTES

5.3.10 Polynomial and Rational Inequalities

This section explains the method for solving polynomial and rational inequalities. Polynomial inequalities can be represented in anyone of the four forms $p(x) > 0$, $p(x) \geq 0$, $p(x) \leq 0$ or $p(x) < 0$, where $p(x)$ is a polynomial. To solve these inequalities we first solve the equation $p(x) = 0$ for each of the zeros of p. These are considered as the endpoints of the solution to the consequent inequality. After finding them, it is easy to determine that between which pairs of endpoints the solutions belong. The following example will make the concept clear.

Example 5.18: Solve the polynomial inequality $x^2 < 4$.

Solution: Take the square root of both sides. Notice that $|x| < 2$ or equivalently, $-2 < x < 2$. Thus, the solution set is the interval $(-2, 2)$, whose graph is shown as follows:



Numerical Line showing Polynomial Inequality $x^2 < 4$

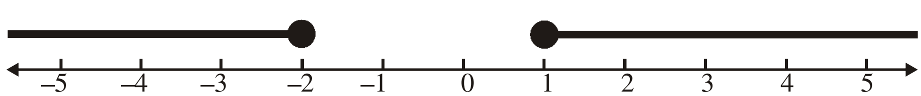
Example 5.19: Solve the polynomial inequality $x^2 + x \geq 2$.

Solution: First solve the corresponding polynomial equation $x^2 + x = 2$ or equivalently $p(x) = x^2 + x - 2 = 0$. Using trial and error, we find that $p(x)$ has the factorization $(x + 2)\ (x - 1)$ implying that its zeros are at -2 and 1 which are the endpoints of the solution set.

Every interval is tested with these endpoints. The first interval is $(-\infty, -2)$. To check that this interval is part of the solution set one of its points is tested. Start with $x = -3$. The left side of the inequality becomes $(-3)^2 - 3 = 6$, which is evidently greater than or equal to -2, hence the inequality holds for this value of x. Hence, it is clear that the interval $(-\infty, -2)$ is part of the solution set.

Now test the interval $(-2, 1)$. By plugging $x = 0$, we check whether inequality $x^2 + x \geq 2$ holds. Evidently it does not, because the left side becomes zero when 0 is substituted for x and 0 is not greater than or equal to 2. Hence, it is clear that the interval $(-2, 1)$ is not part of the solution set.

Finally test the interval $(1, \infty)$. Take the value as $x = 2$. By plugging into the left side of the inequality it becomes $2^2 + 2 = 6 \geq 2$. Hence, it is clear that this interval is part of the solution set. As there are no more endpoints the entire solution set is found, which is $(-\infty, -2) \cup (-1, \infty)$. This can be represented in the graphical form as follows:



Graphical Representation of the Resultant Solution Set

Example 5.20: Solve the polynomial inequality $x^2 - x \leq 1$.

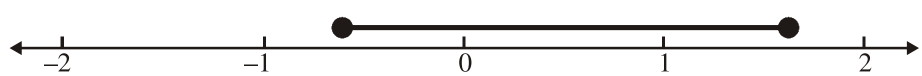
Solution: First solve the corresponding polynomial equation $x^2 - x = 1$ or equivalently $p(x) = x^2 - x - 1 = 0$. By applying the quadratic formula the solutions to this equation are found as $x = (1 \pm \sqrt{5})/2$, which are approximately -0.618 and 1.618. Incidentally, the number $(1 + \sqrt{5})/2$ is an important mathematical constant and is termed as the ***golden mean*** and is denoted by the Greek letter φ.

Test all the three intervals. First test the interval $(-\infty, -0.618)$ by testing $x = -1$. By plugging this into the left side of the inequality we get 2, which is not less than or equal to -1. Hence, it is clear that the solution set does not contain the interval $(-\infty, -0.618)$.

Test the next interval $(-0.618, 1.618)$. To do this test the point $x = 0$. By substituting this into the left side of the inequality we get 0, which is evidently less than or equal to 1. Hence, it is clear that the solution set contains the interval $(-0.618, 1.618)$.

Finally, test the last interval $(1.618, \infty)$. We test the value $x = 2$. Plugging into the left side of the inequality, we obtain 2, which is not less than or equal to -1. Hence, it is clear that the solution set does not contain this interval.

Putting together, we get the solution set which consists of the sole interval $((1 - \sqrt{5})/2, (1 + \sqrt{5})/2) \approx (-0.618, 1.618)$. The following figure shows the graph of this solution set.



Graph of the Solution Set $((1 - \sqrt{5})/2, (1 + \sqrt{5})/2) \approx (-0.618, 1.618)$

Example 5.21: Solve the polynomial inequality $x^3 - 6x^2 + 11x > 6$.

Solution: The corresponding polynomial equation is $p(x) = x^3 - 6x^2 + 11x - 6 = 0$. Using synthetic division find the solutions of this equation which are $x = 1$, $x = 2$ and $x = 3$.

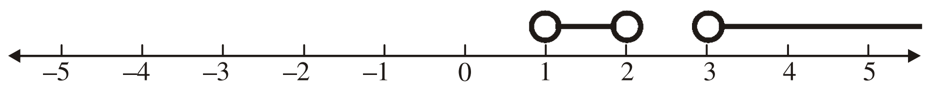
Now there are four intervals for testing. First test the interval $(-\infty, 1)$. The easy point in this interval to text is $x = 0$. Substituting $x = 0$ into the inequality, we get 0 on the left side which is obviously not greater than 6. Hence, it is clear that the interval $(-\infty, 1)$ does not belong to the solution set.

Test the next interval (1, 2). By plugging in $x = 3/2$ to the left side, we get $51/8 = 6\,3/8$, which is greater than 6. Hence, it is clear that the solution set contains the interval (1, 2).

Test the third interval (2, 3). By plugging in $x = 5/2$, we find the left side of the inequality becomes $45/8 = 5\frac{5}{8}$, which is less than 6, whence the inequality does not hold for this point and thus the solution set does not contain the interval (2, 3).

The last interval we must test is $(3, \infty)$. Plugging in $x = 4$ to the left side of the inequality, we obtain 12, which is greater than 6. Thus the solution set contains the interval $(3, \infty)$.

Putting everything together, we see that the solution set is equal to (1, 2) *”(3, ∞). A graph of this solution set is shown in the following figure:



Graph Showing Solution Set is Equal to (1, 2)∪(3, ∞)

In every example we have looked at so far, the intervals we have tested have alternated between belonging and not belonging to the solution set. This is usually but not always the case. Below is an exception to this rule.

Example 5.22: Solve the polynomial inequality $x^3 - 3x < 2$.

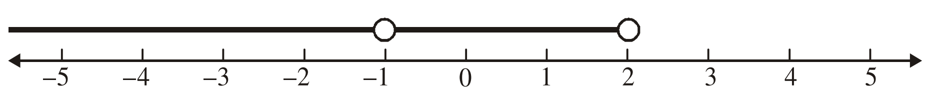
Solution: The corresponding polynomial equation is $p(x) = x^3 - 3x - 2 = 0$. By means of synthetic division, we find that $p(x)$ factors as $(x + 1)^2 (x - 2)$. Thus, the endpoint intervals are $x = -1$ and $x = 2$.

The first interval we must test is $(-\infty, -1)$. Plugging in $x = -2$, we obtain -2 on the left side of the inequality, which is less than 2, whence the solution set contains the interval $(-\infty, -1)$.

The next interval to test is $(-1, 2)$. Plugging in $x = 0$, the left side the inequality becomes 0, which is once again less than 2. Thus, the solution set also contains the interval $(1, 2)$. Note that it does not contain -1, however, since substituting -1 into the left side of the inequality yields 2, which is not less than 2.

The final interval to test is $(2, \infty)$. Plugging in $x = 3$, we see that the left side of the inequality becomes 18, which is not less than 2. Thus, we see that the interval $(2, \infty)$ is not part of the solution set.

Putting everything together, we see that the solution set is $(-\infty, -1)\cup(-1, 2)$. The graph for this is shown as follows:



Graph for Solution Set (– ∞, –1)∪(–1, 2)

It is somewhat difficult to solve rational, but the strategy is the same. To do this, one has to keep track of both the zeros and the singularities of the rational function and proceed in the same way as discussed before.

Example 5.23: Solve the rational inequality $(x^2 - 4) / (3x^2 + 8x - 3) \geq 0$.

Solution: We have already graphed the rational function $f(x) = (x^2 - 4) / (3x^2 + 8x - 3)$, hence we know the values of x which is $f(x) > 0$. Though, it is good to assume that the graph is not accessible. The approach is to first find all zeros and singularities of $f(x)$. Because the numerator factors are $(x + 2)(x - 2)$ and the denominator factors are $(x + 3)(3x - 1)$, we observe that the zeros of f are at -2 and 2 and the singularities are -3 and 1/3. These are the endpoints of the intervals.

NOTES

Test the first interval $(-\infty, -3)$. Because $f(-10) = 96 / 217 > 0$, hence, it is clear that this interval belongs to the solution set.

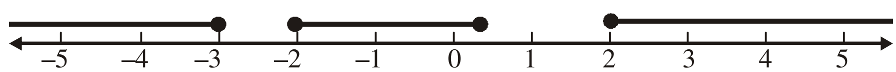
Test the second interval $(-3, -2)$. We have $f(-5/2) = -17/4 < 0$, hence, it is clear that this interval does not belong to the solution set.

Test the third interval $(-2, 1/3)$. Since $f(0) = 4/3 > 0$, hence it is clear that this interval belongs to the solution set.

Now test the fourth interval $(1/3, 2)$. We have $f(1) = -3/8 < 0$, hence it is clear that this interval does not belong to the solution set.

Finally test the last interval $(2, \infty)$. We have $f(10) = 96 / 377 > 0$, hence it is clear that this interval belongs to the solution set.

Putting the findings together, we find the solution set $(-\infty, -3] \cup [-2, 1/3] \cup [2, \infty)$. The graph for the obtained solution set is shown as follows:



Graph for the Obtained Solution Set $(-\infty, -3] \cup [-2, 1/3] \cup [2, \infty)$

CHECK YOUR PROGRESS

5. Can two parts of a complex number be represented graphically?
6. Which is the important key of a function?
7. What do you understand by the singularity of $f(x)$?
8. How can polynomial inequalities be represented?

5.4 USES OF COMPLEX NUMBERS

Complex number finds use in a number of scientific and engineering calculations. While solving an equation, if the solution yields square root of a negative number, there it indicates presence of a complex number. In the field of electrical engineering and telecommunication complex numbers come. Although a complex number involves imaginary number it has correspondence to the real word situation. As you have seen, a complex number can be given a graphical addition in place of direct arithmetic addition.

5.4.1 History of Complex Numbers

Reference for square roots negative real numbers was found in the work of Heron of Alexandria, the Greek during 1st century AD. He felt it by considering volume of the frustum of a pyramid that was considered impossible. These problems of square root of negative number became very prominent by work of Italian mathematicians in finding third and fourth roots of polynomials. Although main interest was in finding the real solution but, they were also interested in manipulating square roots of negative numbers. The term 'imaginary' for such quantities was first coined by Rene Descartes during the period of 17th century. This term was used to mean derogatory. In the period of 18th century, work of Leonhard Euler and Abraham De Moivre established many concepts related to complex numbers. Famous formulae, known as De Moivre's theorem was given on his name.

According to De Moivre's theorem:

$$(\cos\theta + i\sin\theta)^n = \cos n\theta + i\sin n\theta$$

Where θ is any complex number or real number and n is an integer.

This formula is considered very important as it connects a complex numbers to trigonometry. The expression '$\cos\theta + i\sin\theta$' is also known as 'cis θ' in short.

We may expand LHS of the De Moivre's formula and after comparing both the parts (assuming that θ is a real number), expressions can be derived for $\cos(n\theta)$ and $\sin(n\theta)$ involving terms of $\cos(\theta)$ and $\sin(\theta)$. Also, this formula can be used for finding nth roots of unity. We can find a complex number z such that $z^n = 1$.

Existence of complex numbers was accepted only after its geometrical interpretation was given by Caspar Wessel in the year 1799. Same thing was discovered later after several years. This was popularized by Gauss (Carl Friedrich Gauss) and due to his efforts complex numbers theory got further expansion.

Wessel's memoir was given in Proceedings of the Copenhagen Academy in 1799. This work is complete and very clear in content that can be compared to any modern work. He considered sphere, and provided a quaternion theory that can develop spherical trigonometry. In the year 1806 Argand got the same idea that was suggested by Wallis and also issued a paper on this subject. Argand's essay provided a base on which present day graphic presentation of complex numbers works. In 1832 Gauss published his work on it and brought it in mathematical world, giving it a prominence. Mention should also be made of the excellent small treatise presented by Mourey in the year 1828. In this period foundations were laid scientifically for the theory of directional numbers.

Different terms were used by different mathematician in relation to complex number. Argand called $\cos\theta + i\sin\theta$ the *direction factor*, and the *modulus*. Cauchy, in the year 1828 called the same as the *reduced form* and Gauss used i for square root of -1 and (imaginary number) and used the term *complex number* for a binomial term $a + bi$, and named $a^2 + b^2$ as the *norm*. Hankel, in the year 1867, used *direction coefficient*, for expression '$\cos\theta+i\sin\theta$'. The term *absolute value,* for *modulus,* was used by Weierstrass.

After Cauchy and Gauss, many names came as high ranking contributors. Of these, following names are worth mention:

Contribution was made in 1844 by Kummer, in 1845 by Kronecker, Leopold and Peacock and Scheffler in 1845, 1851 and 1880 and Bellavitis in 1835 and 1852.

NOTES

Also, Scheffler (1845, 1851, 1880), Bellavitis (1835, 1852), Peacock (1845), and De Morgan (1849) were great contributors. Name of Möbius is also worth mention for his numerous works on geometric applications of complex numbers. The work of Dirichlet helped expansion of this theory that included complex numbers for primes, congruences, reciprocity, and like that as was for real numbers.

A set of complex number, closed under addition, subtraction and multiplication, is known as complex ring or field. Study was made for complex numbers having form $a + bi$. [a and b being integral, or rational and i is here either of the two roots of $x^2 + 1 = 0$]. Ferdinand Eisenstein who was his student, carried study of the type $a + b\omega$, where ω is the complex cubic root (complex root of $x^3 - 1 = 0$). There were other classes too for such cases and these are known as cyclotomic fields for complex numbers. These have been derived from equation $x^k - 1 = 0$, by finding roots of unity in case of higher values of k. Such generalization was mainly due to the work Kummer, who invented ideal numbers, expressed as geometrical entities by Felix Klein in 1893. General theory for fields had been created by Évariste Galois. He carried out study on fields that was generated by roots of a polynomial equation in one variable.

Complex numbers, thus have a field, known as complex number field and is denoted by **C.** A complex number $x + iy$, is denoted as an ordered pair of (x, y). A number whose imaginary part is zero is a real number. A number, whose real part is zero, is a purely imaginary number and a number whose neither part is zero is a complex number. Viewing this way every real number is a subset of complex number. Or stated in terms of field one may say that **R,** the field of real numbers is a subfield of **C.** Thus, we identify a real number x as complex number $(x, 0)$ and an imaginary number may be denoted as complex number $(0, y)$.

In the field of complex number **C**, we have:

Additive identity as (0,0), multiplicative identity as (1,0), additive inverse of (x, y) as $(-x, -y)$

Thus, complex field **C** can also be defined as the topological closure of algebraic numbers and the algebraic closure of **R**.

5.4.2 Geometrical Representation of Conjugates

NOTES

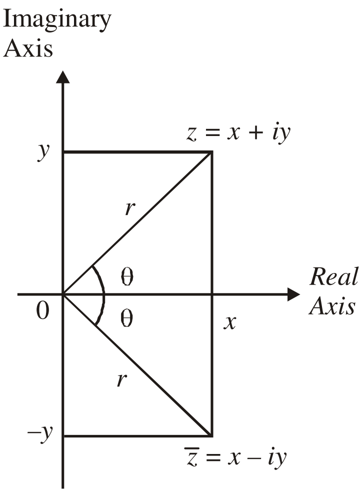


Figure 5.8 *Geometrical Representation of Conjugate*

When both the parts are positive the point falls in the first quadrant of the plane. A conjugate of a complex number is found by changing the sign of its imaginary part (Figure 5.8). Thus, conjugate of $x + yi$ is $x - yi$ and it falls in fourth quadrant and is the mirror image of $x + yi$. Its reverse is also true. The complex number $x - yi$ is the conjugate of $x + yi$.

Both these complex numbers are known as conjugate pairs.

Thus, a complex number is viewed as a point on two dimensional Cartesian coordinate system. Speaking in terms of vector, it shows a position vector. Such a representation is also known as **Argand diagram**.

One may view addition of two complex numbers as addition of two vectors. Multiplication with a fixed complex number may be viewed as simultaneous rotation and stretching.

Also, multiplication with i signifies 90 degrees counter clockwise rotation. The geometric interpretation of $i^2 = -1$ is a sequence of two rotations of 90 degree resulting in rotation of 180 degree. The fact, $(-1) \times (-1) = +1$ can be interpreted as combination of two rotations of 180 degree each.

5.4.3 Geometric Interpretation of Addition and Multiplication Operations

Operations of addition and multiplication have already been described algebraically above. The same can be shown on an Argand plane as Argand diagram (Figures 5.9 and 5.10).

Suppose A and B are two complex numbers shown as points on Argand plane. Let X be the addition of these two and hence, $X = A + B$. O is the origin. Summation of these two complex number on Argand diagram shows two congruent triangles having vertices 0, A, B, and X, B, A. Thus, addition of two complex numbers and addition of two vectors are same. A complex number on an Argand plane may also be taken as a vector and its position as position vector.

NOTES

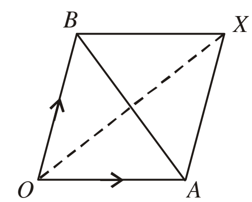


Figure 5.9 *Graphical Addition*

Multiplication of two complex numbers can also be shown graphically by representing them on Argand plane. We take same two complex number A and B and let X be their product such that $X = AB$. Product of two complex number is also a complex number and hence, this can also be shown on the Argand plane. Let this point be shown as X. Let O be the origin. Hence, **product** of A and B is a point X such that triangles having vertices 0, 1, A, and 0, B, X, for similar triangles.

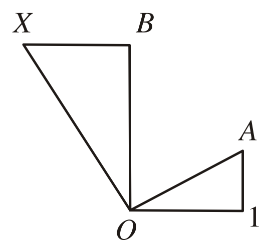


Figure 5.10 *Graphical Multiplication*

Such geometric interpretations translate algebraic problems into geometrical problem and geometric problems can also be analysed algebraically. For example,

Absolute value, conjugation and distance

We recall that *absolute value* (or *modulus* or *magnitude*) of a complex number is given by $|z| = \sqrt{(a^2 + b^2)}$, if $z = a + ib$. When $z = r(\cos\theta + i\sin\theta) = re^{i}\theta$, it is defined as $|z| = r$.

Absolute value can be used to find distance between two points. If z and w are two complex numbers, then distance between then is given by distance function $d(z, w) = |z - w|$. Equation for regular geometrical figures such as straight lines, circles and conic sections can also be written in terms of complex numbers. Addition, subtraction, multiplication and division involving complex numbers are continuous operations.

The *complex argument* of $z = re^{i}\theta$ is θ. It is is unique modulo 2π. That means that any two values of complex argument θ always differ by an integral multiple of 2π.

Square Root of a Complex Number

Square root of a complex number is also a complex number.

Let, $z = x + iy$ be a complex number.

Square root of z, i.e., $\sqrt{z}$ will also be a complex number. Let $a + ib = \sqrt{z}$.

Value of a and b, the square root is found.

Squarring both the sides, we get,

NOTES

$(a + ib)^2 = x + iy$

$\Rightarrow \quad a^2 + i\,2ab + z^2b^2 = x + iy$

$\Rightarrow \quad a^2 - b^2 + i\,2ab = x + iy$

Equating real and imaginary part together, we get,

$a^2 - b^2 + i\,2ab = x + iy$

Equating real and imaginary part together, we get,

$$a^2 - b^2 = x \qquad ...(5.1)$$

$$2\,ab = y \qquad ...(5.2)$$

We know that,

$(a^2 + b^2)^2 - 4a^2b^2 = (a^2 - b^2)^2$

$\Rightarrow \quad (a^2 + b^2)^2 - (2ab)^2 = x^2$

$\Rightarrow \quad (a^2 + b^2)^2 - y^2 = x^2$

$(a^2 + b^2)^2 = x^2 + y^2$

$$a^2 + b^2 = \pm\sqrt{x^2 + y^2} \qquad ...(5.3)$$

Adding equations (1) and (3) we get,

$$2a^2 = x \pm \sqrt{x^2 + y^2}$$

subtracting (1) from (3) we get,

$$2b^2 = \pm\sqrt{x^2 + y^2} - x$$

We get the value of a and b. Hence square root of $x + iy$ can be found.

Example 5.24: Find square root of a complex number given by $z = -3 + 4i$

Solution: Let $x + iy = \pm\sqrt{z} = \pm\sqrt{-3+4i}$

Squaring both the sides,

$$(x + iy)^2 = -3 + 4i$$

$\Rightarrow \quad x^2 + i2xy + i^2y^2 = -3 + 4i$

$\Rightarrow \quad x^2 - y^2 + i\,2xy = -3 + 4i$

Equating real and imaginary parts

$$x^2 - y^2 = -3 \qquad ...(1)$$

$$2xy = 4 \qquad ...(2)$$

$$(x^2 + y^2)^2 = (x^2 - y^2)^2 + 4x^2y^2$$

$$= (-3)^2 + (4)^2 = 9 + 16 = 25$$

$$\Rightarrow \quad (x^2 + y^2)^2 = 25$$

$$x^2 + y^2 = \pm 5 \qquad ...(3)$$

NOTES

Adding equations (1) and (3), we get

$$2x^2 = -3 \pm 5 = 2 \text{ or } -8$$

$$\Rightarrow \quad x = \pm 1 \text{ or } \pm 2i$$

Since x is a real number, we take $x = \pm 1$, and discard imaginary value.

$$2xy = 4 \text{ given}$$

$$y = \frac{4}{2x} = \frac{4}{2\times(\pm 1)} = \pm 2.$$

When $x = 1$, $y = 2$,

$x = -1$, $y = -2$

We get square root of $-3 + 4i$

as

(1) $1 + 2i$,

(2) $-1 - 2i$,

Hence, square root of $-3 + 4i$

is $\pm (1 + 2i)$.

Example 5.25: Find square root of a complex number given as:

$$z = 7 + 24i$$

Solution: Let

$$x + iy = \pm \sqrt{7+24i}$$

Squarring both the sides,

$$x^2 + i2xy + i^2y^2 = \sqrt{7+24i}$$

$$\Rightarrow \quad x^2 - y^2 + i\,2xy = 7 + 24i$$

$$x^2 - y^2 = 7 \qquad ...(1)$$

$$2xy = 24 \qquad ...(2)$$

$$(x^2 + y^2)^2 = (x^2 - y^2)^2 + 4x^2y^2$$

$$= (x^2 - y^2)^2 + (2xy)^2$$

$$= (7)^2 + (24)$$

$$= 49 + 576$$

$$= 625$$

$$x^2 + y^2 = \pm 25 \qquad ...(3)$$

Adding equations (1) and (3), we get

$$2x^2 = 7 \pm 25$$
$$2x^2 = -32\ x\ -18$$
$$x^2 = 16 \text{ or } -9$$

We take $x^2 = 16$ since x is a real number.

Hence, $x = \pm 4$

From equation (2) we get the value of $y = \frac{24}{2\times x} = \frac{24}{2\times(\pm 4)} = \pm 3$,

Hence, square root of $7 + 24i$ is $\pm (4 + 3i)$.

5.4.4 Matrix Representation of Complex Numbers

Complex number can be represented in matrix form. Every complex number may be put as 2×2 matrix having entries for real numbers that stretches and rotates points of the plane. Every such matrix has the form

$\begin{bmatrix} a & -b \\ b & a \end{bmatrix}$ where $a, b \in$ R (set of real numbers). Since sum and product of two matrices are complex numbers, this can also be represented in this form. A matrix that is non-zero is invertible, and such inverse also has this form. Thus, matrices having such form denote a field. Such matrices may be written as:

$$\begin{bmatrix} a & -b \\ b & a \end{bmatrix} = a\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + b\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$$

This form of matrix suggests that the real number 1 can be identified with an identity matrix $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ and an imaginary unit number i with $\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$ that is rotated by a positive 90 degrees rotation (counter-clockwise). Square of such a 2×2 matrix has a determinant value of -1.

In a similar way absolute value of a complex number can also be shown as a determinant value of the matrix that expresses a complex number. If a complex number z, is expressed as matrix this equals square root of the determinant value as shown below: $|z|^2 = \begin{vmatrix} a & -b \\ b & a \end{vmatrix} = (a^2) - ((-b)(b)) = a^2 + b^2$

Even transformation of a plane can also be shown. Such transformation causes rotation of points by an angle that equals the argument of the complex number, having length multiplied by a factor that equals the absolute value of the complex number. Viewing this way conjugate of a complex number z denotes rotation by same angle as that of z with real axis, but in opposite direction and its length also scales in the same manner as z. This operation corresponds to transpose of a matrix that corresponds to z.

In case a matrix has complex numbers instead of real as elements of the given matrix resulting algebra is that of the quaternions.

5.5 PROPERTIES OF COMPLEX NUMBERS

NOTES

Real Vector Space

As you have seen, a complex number also represents a vector and a set of all such complex number, denoted by **C**, is a real vector space in two dimensions. Ordering as found with real numbers does not apply to complex numbers in case of arithmetic addition and hence **C** cannot have an ordered field. As a generalized statement, a field having imaginary number cannot be ordered.

Mapping of **R**-linear function **C** → **C** assumes the general form as given below: $f(z) = az + b\overline{z}$

Here, *a* and *b* are complex coefficients and can be written symbolically as $a, b \in$ **C**.

Here, function given as: $f(z) = az$ denotes rotations that is also combined with scaling, But the function $f(z) = b\overline{z}$ denotes reflections that are combined with scaling.

Solutions of Polynomial Equations

A *root* of a polynomial p is a complex number z satisfying the equation $p(z) = 0$. An n-degree polynomials having coefficients real or complex, has exactly n complex roots. This is the Fundamental Theorem of Algebra, and this also tells that complex numbers are defined for an algebraically closed field.

Complex Analysis

The study of functions of a complex variable is known as complex analysis and has enormous practical use in applied mathematics as well as in other branches of mathematics. Often, the most natural proofs for statements in real analysis or even number theory employ techniques from complex analysis (see prime number theorem for an example). Unlike real functions which are commonly represented as two dimensional graphs, complex functions have four dimensional graphs and may usefully be illustrated by color coding a three dimensional graph to suggest four dimensions, or by animating the complex function's dynamic transformation of the complex plane.

5.5.1 Applications of Complex Numbers

Complex number finds various applications in many fields of science and engineering. It is widely used in the field of control theory in electrical engineering and telecommunication .

Control Theory

In most of the control theory in the field of telecommunication, transformation is done from the domain of time to that of frequency. Laplace transform is used in the field of electrical engineering for such transformations. Poles and zeros of the system are analyzed in *complex plane*. Complex plain is used in analysis of the Nyquist plot, root locus and Nichols plot techniques.

Position of poles and holes are of special importance in the root locus method. Situations, such as position of poles and holes with respect of quadrants is especially important. These may lie in the left and right half planes. This is same as knowing whether real part is greater than or less than zero. If a system has poles:

NOTES

In right half of the plane - it is unstable,

All in left half of the plane - it is stable,

Falling on imaginary axis – it is marginally stable.

Also, if a system contains zeros in right half of plane, it is called non minimum phase system.

5.5.2 Signal Analysis

In the filed of electrical engineering and telecommunication, signal analysis is done using complex numbers to describe signals that vary periodically. Absolute value or modulus of $|z|$ corresponds to the amplitude and argument $\arg(z)$ as phase of a sine wave having some given frequency.

Fourier transform and Fourier analysis is done for writing a given signal having real value as a sum of many periodic functions, often written as the part having real of complex functions of the form $f(\varepsilon) = ze^{iwt}$. Here, ω denotes angular frequency that is equal to $2\pi f$, where f denotes frequency. Complex number z does encoding of phase and amplitude.

In the fields of electrical engineering, such technique is utilized by varying currents and voltages. In electrical systems, resistances offered by resistors are taken as real valued and those offered by capacitors and inductors are taken as imaginary. These three that offer combined resistance as known by another name, impedance. Resistances offered by inductors and capacitors are known by another name reactance.

If impedance is denoted by Z as combination of resistance R, and combined reactance of inductor and capacitor X, then Z is denoted as a complex number as:

$Z = R + i\,X.$

Electrical engineering texts use j in place of i for showing imaginary quantity.

Improper Integrals

Complex number is used in the field of applied sciences for finding improper integrals that are real-valued, and complex-valued functions are used for this. There are several methods to do the same.

Quantum Mechanics

In the field of quantum mechanics also complex numbers are of importance. The underlying theory has been built on Hilbert spaces (a space having infinite dimensions) over **C**.

Relativity

Taking time variable as imaginary use of formulas for finding metric on space time becomes simpler.

NOTES

Applied Mathematics

In solution of differential equations all complex roots r of characteristic equation is found first for a linear differential equation. Attempts to solve for the base function is made after this using base functions on the form $f(t) = e^{rt}$.

Fluid Dynamics

Complex functions, in fluid dynamics are used for describing potential flow in two dimensions.

Fractals

Some fractals known as Julia set and Mandelbrot set are plotted in the complex plane.

CHECK YOUR PROGRESS

9. What is a complex ring or field?
10. What is complex analysis?

5.6 SUMMARY

In this unit, you have learned that:

- Girolamo Cardano was an Italian mathematician who first realized the need of finding complex numbers. While solving cubic equations, he came across expressions that contained square root of a negative number. Thus, the concept of an imaginary number took birth. However, this was accepted as a part of mathematical concept by the work of Abraham De Moivre, Bernoullis and Leonhard Euler, who carried out studies on this topic in detail. De Moivre was a French-English mathematician, Bernoullis were mathematicians in a Swiss family and Euler was also a Swiss mathematician. This term 'imaginary' was used by Rene Descartes in 17th century.
- If we define that the square of imaginary number i is -1, then it is possible to find solution of $x^2 + 1 = 0$. In mathematics, one can describe the notions if they satisfy a specific set of logically constant axioms. For example, the defined number i can also be written as $\sqrt{-1}$, where the real numbers can be included to extend it along with imaginary numbers. The simple method to perform is to describe the set of complex numbers as the set of all numbers of form $z = x + yi$, where x and y are arbitrary real numbers. Here, number x is termed as the real part and y as the imaginary part of the complex number z.

- Basically, there are two functional operations on complex numbers. First is termed as *complex conjugation* and the second is the *modulus*, also famous as the *complex norm* or *absolute value*. For a given complex number $z = x + yi$, one can describe the *complex conjugate* of z (written as z) as the complex number $z = x - yi$ such that in the complex plane, the point corresponding to z is $(x, -y)$.
- In a complex number, when the real part and its imaginary part are equal to zero, then only the complex number is equal to zero. For example, $a + bi = 0$ if and only if $a = 0$ and $b = 0$. Similarly, the complex numbers are equal if their real and imaginary parts are equal.
- A complex number has two parts, real and imaginary. A number is imaginary when it is the square root of a negative real number. For example, if $x^2 = 1$, then $x = \pm\sqrt{1} = \pm 1$. Such a square root has both real values. But if there is an equation $x^2 + 1 = 0 \rightarrow x^2 = -1$ and hence, $x = \sqrt{-1}$ and this number is imaginary since there is no real number that satisfies this condition, then this number, given by $\sqrt{-1}$, is an imaginary number and it is designated by using letter i.
- *Modulus* of a complex number is a length of vector *OP* which represents the complex number in a coordinate *complex* plane.
- Two parts of a complex number can be represented graphically by representing one along the real axis (shown horizontally) and the other along imaginary axis, at positive right angle to it. This is like presenting a point on a Cartesian plane by an ordered pair (x, y).
- An Argand plane is a way to represent a complex number as points on rectangular coordinate plane. This also known as *complex plane* in which x-axis is used for real axis and y as imaginary axis. Argand plane is so named as an amateur mathematician named Jean Robert Argand described the plane in his paper in the year 1806.
- On an Argand plane, the position of a point can be shown both in rectangular coordinate system as well as polar coordinate system. A complex number z has been shown as an ordered pair (x, y) on a Cartesian plane and as $(r, i\theta)$ in polar coordinates.
- The quadratic equations are of the form $ax^2 + bx + c = 0$, where a, b and c are considered as real-valued constants along with a non-zero. The left part of this equation is a quadratic function of the form $f(x) = ax^2 + \text{b}x + c$ and the graph of this quadratic function is termed as parabola, which is a special type of curve. The simple parabola is specified by the function $\text{f}(x) = x^2$.
- The complexity of the graph depends on the degrees of polynomials growth. A polynomial of degree n will have up to n real zeros so that its graph crosses the x-axis n times. However, one can easily find the zeros of some polynomials by simply plotting sufficient points. Reasonably exact graphs can be made of general polynomials.

NOTES

- In synthetic division, we just write the rows of numbers and not the powers of the variable x. In the first/top row, the coefficients of the dividend are written which is preceded to the left by the negative of the constant coefficient of the divisor divided by the linear coefficient.
- Finding zeros of polynomial function is the important key of a function, termed as roots. These are the values of the variable x for $f(x) = 0$. Principally, it is extremely useful in finding the zeros of a univariate polynomial, whereas to find the zeros of linear polynomials is completely trivial.
- Rational functions have the common form $f(x) = p(x)/q(x)$, where p and q are polynomials. When $q(x)$ is zero and $p(x)$ is not, $f(x)$ remains undefined. This results as singularity of $f(x)$. For values of x close to the singular value, $|f(x)|$ becomes very large. Hence, while graphing rational functions first check its singularities.
- Polynomial inequalities can be represented in any one of the four forms, $p(x) > 0$, $p(x) \geq 0$, $p(x) \leq 0$ or $p(x) < 0$, where $p(x)$ is a polynomial. To solve these inequalities, we first solve the equation $p(x) = 0$ for each of the zeros of p. These are considered as the endpoints of the solution to the consequent inequality. After finding them, it is easy to determine that between which pairs of endpoints the solutions belong.
- Complex number finds use in a number of scientific and engineering calculations. While solving an equation, if the solution yields square root of a negative number, it indicates the presence of a complex number. Although a complex number involves imaginary number, it has correspondence to the real, word situation. As you have seen, a complex number can be given a graphical addition in place of a direct arithmetic addition.
- A complex number also represents a vector and a set of all such complex numbers, denoted by C, is a real vector space in two dimensions. Ordering as found with real numbers does not apply to complex numbers in case of arithmetic addition and hence C cannot have an ordered field. As a generalized statement, a field having imaginary number cannot be ordered.
- The study of functions of a complex variable is known as complex analysis. It has enormous practical use in applied mathematics as well as in other branches of mathematics.

5.7 KEY TERMS

- **Imaginary number:** It is the square root of a negative real number.
- **Binomial number:** It is a number with an expression containing two terms.
- **Argand plane:** It is a way to represent a complex number as points on a rectangular coordinate plane.

5.8 ANSWERS TO 'CHECK YOUR PROGRESS'

NOTES

1. The quadratic equation $x^2 + 1 = 0$ can also be written as $x^2 = -1$.
2. According to the rule for zero, when the real part and the imaginary part of a complex number are equal to zero, then only the complex number is equal to zero.
3. The two parts of a complex number are real and imaginary.
4. The arithmetic operations that can be performed easily on complex numbers are addition and subtraction and the ones which are complex are multiplication and division.
5. Yes, two parts of a complex number can be represented graphically.
6. Finding zeros of polynomial function is the important key of a function.
7. Rational functions have the common form $f(x) = p(x)/q(x)$, where p and q are polynomials. Every time $q(x)$ is zero and $p(x)$ is not, $f(x)$ remains undefined. This results as singularity of $f(x)$.
8. Polynomial inequalities can be represented in any one of the four forms $p(x) > 0$, $p(x) \geq 0$, $p(x) \leq 0$ or $p(x) < 0$, where $p(x)$ is a polynomial.
9. A set of complex number, closed under addition, subtraction and multiplication, is known as complex ring or field.
10. The study of functions of a complex variable is known as complex analysis.

5.9 QUESTIONS AND EXERCISES

Short-Answer Questions

1. Write a short note on imaginary numbers.
2. Find the value of i^{393}.
3. Find $Z_1 \times Z_2$ when $Z_1 = 3 + i$ and $Z_2 = 7 + 3i$
4. Find the square root of $3 + 4i$.
5. Write a short note on Argand diagram.
6. Find Z_1/Z_2 when $Z_1 = 4 + 3i$ and $Z_2 = 2 + 5i$
7. What is complex number rotation of a point (2, 9) on an Argand plane?
8. Find the modulus of Z, when $Z = 12 + 5i$.
9. Find the argument of the complex number $z = 12 + 5i$.
10. Write a short note on signal analysis.

NOTES

Long-Answer Questions

1. Explain the general rules for performing operations on complex numbers.
2. Graphically represent the polynomial $f(x) = x^4 - 10x^2 + 9$.
3. Explain the zeros of polynomial functions.
4. Explain the operations of addition and multiplication and on an Argand plane.
5. Discuss the matrix representation of complex numbers.
6. Graphically represent the function $f(x) = 1/(1 - x^2)$.
7. Solve the rational inequality $(x^2 - 4) / (3x^2 + 8x - 3) \geq 0$.
8. Find square root of i.
9. A complex number is given by $6 + 8i$. Find its modulus and argument.
10. Represent the complex number $6 + 8i$ in trigonometric and exponential forms.

5.10 FURTHER READING

Khanna, V.K, S.K. Bhambri, C.B. Gupta and Vijay Gupta. *Quantitative Techniques*. New Delhi: Vikas Publishing House.

Khanna, V.K, S.K. Bhambri and Quazi Zameeruddin. *Business Mathematics.* New Delhi: Vikas Publishing House.

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